Chapter 12
Introduction to Three Dimensional Geometry

Exercise 12.1

Question 1:

Find the coordinates of the point which divides the line segment joining the points (–2, 3, 5) and (1, –4, 6) in the ratio
(i) 2:3 internally, (ii) 2:3 externally.

Solution 1:

(i) The coordinates of point R that divides the line segment joining points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) internally in the ratio m: n are

$$\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}, \frac{mz_2+nz_1}{m+n}\right)$$

Let R (x, y, z) be the point that divides the line segment joining points (–2, 3, 5) and (1, –4, 6) internally in the ratio 2:3

$$x=\frac{2(1)+3(-2)}{2+3},\ y=\frac{2(-4)+3(3)}{2+3}, \text{ and } z=\frac{2(6)+3(5)}{2+3}$$

i.e., $x=\frac{-4}{5},\ y=\frac{1}{5}$, and $z=\frac{27}{5}$

Thus, the coordinates of the required point are $\left(\frac{-4}{5}, \frac{1}{5}, \frac{27}{5}\right)$

(ii) The coordinates of point R that divides the line segment joining points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ externally in the ratio m: n are

$$\left(\frac{mx_2+nx_1}{m-n}, \frac{my_2+ny_1}{m-n}, \frac{mz_2+nz_1}{m-n}\right)$$

Let R (x, y, z) be the point that divides the line segment joining points (–2, 3, 5) and (1, –4, 6) externally in the ratio 2:3

$$x=\frac{2(1)-3(-2)}{2-3},\ y=\frac{2(-4)-3(3)}{2-3}, \text{ and } z=\frac{2(6)-3(5)}{2-3}$$

i.e., x = –8, y = 17, and z = 3

Thus, the coordinates of the required point are (–8, 17, 3).

Question 2:

Given that P (3, 2, –4), Q (5, 4, –6) and R (9, 8, –10) are collinear. Find the ratio in which Q divides PR.

Solution 2:

Let point Q (5, 4, –6) divide the line segment joining points P (3, 2, –4) and R (9, 8, –10) in the ratio k:1.

Therefore, by section formula,

$$(5,\ 4,\ -6)=\left(\frac{k(9)+3}{k+1}, \frac{k(8)+2}{k+1}, \frac{k(-10)-4}{k+1}\right)$$

$$\Rightarrow \frac{9k+3}{k+1}=5$$

$$\Rightarrow 9k+3=5k+5$$

$$\Rightarrow 4k=2$$

$$\Rightarrow k=\frac{2}{4}=\frac{1}{2}$$

Thus, point Q divides PR in the ratio 1:2.

Question 3:

Find the ratio in which the YZ-plane divides the line segment formed by joining the points (–2, 4, 7) and (3, –5, 8).

Solution 3:

Let the YZ plane divide the line segment joining points (–2, 4, 7) and (3, –5, 8) in the ratio k:1.

Hence, by section formula, the coordinates of point of intersection are given by

$$\left(\frac{k(3)-2}{k+1},\frac{k(-5)+4}{k+1},\frac{k(8)+7}{k+1}\right)$$

On the YZ plane, the x-coordinate of any point is zero.

$$\Rightarrow \frac{3k-2}{k+1}=0$$

$$\Rightarrow 3k-2=0$$

$$\Rightarrow k=\frac{2}{3}$$

Thus, the YZ plane divides the line segment formed by joining the given points in the ratio 2:3.

Question 4:

Using section formula, show that the points A (2, –3, 4), B (–1, 2, 1) and C $\left(0,\frac{1}{3},2\right)$ are collinear.

Solution 4:

The given points are A (2, –3, 4), B (–1, 2, 1), and C $\left(0,\frac{1}{3},2\right)$

Let P be a point that divides AB in the ratio k:1.

Hence, by section formula, the coordinates of P are given by

$$\left(\frac{k(-1)+2}{k+1},\frac{k(2)-3}{k+1},\frac{k(1)+4}{k+1}\right)$$

Now, we find the value of k at which point P coincides with point C.

By taking $\frac{-k+2}{k+1}=0$, we obtain k = 2.

For k = 2, the coordinates of point P are $\left(0,\frac{1}{3},2\right)$

i.e., $\left(0,\frac{1}{3},2\right)$ is a point that divides AB externally in the ratio 2:1 and is the same as point P.

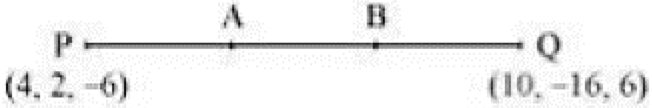
Hence, points A, B, and C are collinear.

Question 5:

Find the coordinates of the points which trisect the line segment joining the points P (4, 2, –6) and Q (10, –16, 6).

Solution 5:

Let A and B be the points that trisect the line segment joining points P (4, 2, –6) and Q (10, –16, 6)



Point A divides PQ in the ratio 1:2. Therefore, by section formula, the coordinates of point A are given by

$$\left(\frac{1(10)+2(4)}{1+2}, \frac{1(-16)+2(2)}{1+2}, \frac{1(6)+2(-4)}{1+2}\right)=(6,-4,-2)$$

Point B divides PQ in the ratio 2:1. Therefore, by section formula, the coordinates of point B are given by

$$\left(\frac{2(10)+1(4)}{2+1}, \frac{2(-16)+1(2)}{2+1}, \frac{2(6)+1(-4)}{2+1}\right)=(8,-10,2)$$

Thus, (6, –4, –2) and (8, –10, 2) are the points that trisect the line segment joining points P (4, 2, –6) and Q (10, –16, 6).

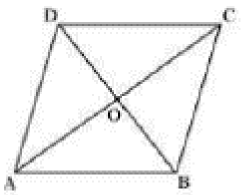
Miscellaneous Exercise

Question 1:

Three vertices of a parallelogram ABCD are A (3, -1, 2), B (1, 2, -4) and C (-1, 1, 2). Find the coordinates of the fourth vertex.

Solution 1:

The three vertices of a parallelogram ABCD are given as A (3, -1, 2), B (1, 2, -4), and C (-1, 1, 2). Let the coordinates of the fourth vertex be D (x, y, z).



We know that the diagonals of a parallelogram bisect each other.
Therefore, in parallelogram ABCD, AC and BD bisect each other.
∴Mid-point of AC = Mid-point of BD

$$\Rightarrow\left(\frac{3-1}{2},\frac{-1+1}{2},\frac{2+2}{2}\right)=\left(\frac{x+1}{2},\frac{y+2}{2},\frac{z-4}{2}\right)$$

$$\Rightarrow(10,2)=\left(\frac{x+1}{2},\frac{y+2}{2},\frac{z-4}{2}\right)$$

$$\Rightarrow\frac{x+1}{2}=1,\frac{y+2}{2}=0, and\ \frac{z-4}{2}=2$$

$\Rightarrow$ x = 1, y = 2 and z = 8

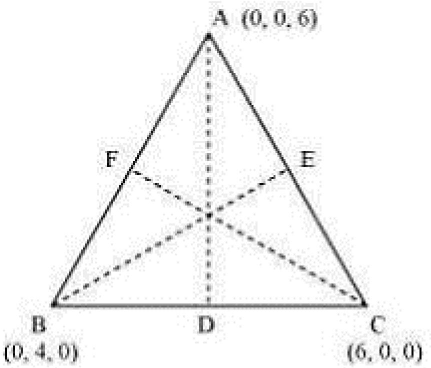
Thus, the coordinates of the fourth vertex are (1, -2, 8).

Question 2:

Find the lengths of the medians of the triangle with vertices A (0, 0, 6), B (0, 4, 0) and (6, 0, 0).

Solution 2:

Let AD, BE, and CF be the medians of the given triangle ABC



Since AD is the median, D is the mid-point of BC

Coordinates of point D = $\left(\frac{0+6}{2},\frac{4+0}{2},\frac{0+0}{2}\right)=(3,2,0)$

$$AD=\sqrt{(0-3)^2+(0-2)^2\ +(6-0)^2}\ =\sqrt{9+4+36}=\sqrt{49}=7$$

Since BE is the median, E is the mid-point of AC.

Coordinate of point E = $\left(\frac{0+6}{2},\frac{0+0}{2},\frac{0+0}{2}\right)=(3,0,3)$

$$BE=\sqrt{(3-0)^2+(0-4)^2+(3-0)^2}=\sqrt{9+16+9}=\sqrt{34}$$

Since CF is the median, F is the mid-point of AB

Coordinates of point F = $\left(\frac{0+0}{2},\frac{0+4}{2},\frac{6+0}{2}\right)=(0,2,3)$

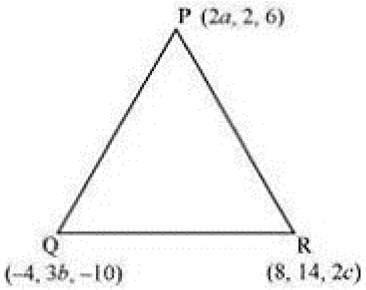
Length of CF $=\sqrt{(6-0)^2\ +(0-2)^2+(0-3)^2}\ =\sqrt{36+4+9}=\sqrt{49}=7$

Thus, the lengths of the medians of ΔABC are 7, $\sqrt{34}$, and 7.

Question 3:

If the origin is the centroid of the triangle PQR with vertices P (2a, 2, 6), Q (-4, 3b, -10) and R (8, 14, 2c), then find the values of a, b and c.

Solution 3:



It is known that the coordinates of the centroid of the triangle, whose vertices are $(x_1, y_1, z_1)(x_2, y_2, z_2)$ *and* (x_3, y_3, z_3), are

$$\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}$$

Therefore, coordinates of the centroid of

$$\Delta PQR = \left(\frac{2a-4+8}{3}, \frac{2+3b+14}{3}, \frac{6-10+2c}{3}\right) = \left(\frac{2a+4}{3}, \frac{3b+16}{3}, \frac{2c-4}{3}\right)$$

It is given that origin is the centroid of ΔPQR

$$\Rightarrow \frac{2a+3}{3} = 0, \frac{3b+16}{3} = 0 \, and \, \frac{2c-4}{3} = 0$$

$$\Rightarrow a = -2, b = \frac{16}{3} \, and \, c = 2$$

Thus, the respective values of a, b and c are $-2, -\frac{16}{3}$ and 2

Question 4:

Find the coordinates of a point on y-axis which are at a distance of $5\sqrt{2}$ from the point P (3, -2, 5).

Solution 4:

If a point is on the y-axis, then x-coordinate and the z-coordinate of the point are zero.

Let A (0, b, 0) be the point on the y-axis at a distance of $5\sqrt{2}$ from point P (3, -2, 5).

Accordingly, AP = $5\sqrt{2}$

$\Rightarrow AP^2 = 50$

$\Rightarrow (3-0)^2 + (-2-b)^2 + (5-0)^2 = 50$

$\Rightarrow 9+4+b^2+4b+25=50$

$\Rightarrow b^2+4b-12=0$

$\Rightarrow b^2+6b-2b-12=0$

$\Rightarrow (b+6)(b-2)=0$

$\Rightarrow b=-6 \, or \, 2$

Thus, the coordinate of the required points are (0, 2, 0) and (0, -6, 5)

Question 5:

A point R with x-coordinate 4 lies on the line segment joining the points P (2, -3, 4) and Q (8, 0, 10). Find the coordinates of the point R.

[*Hint*: suppose R divides PQ in the ratio k: 1.] The coordinates of the point R are given by

Solution 5:

The coordinates of points P and Q are given as P (2, -3, 4) and Q (8, 0, 10). Let R divide line segment PQ in the ratio k:1.

Hence, by section formula, the coordinates of point R are given by

$$\left(\frac{k(8)+2}{k+1}, \frac{k(0)-3}{k+1}, \frac{k(10)+4}{k+1}\right) = \left(\frac{8k+2}{k+1}, \frac{-3}{k+1}, \frac{10k+4}{k+1}\right)$$

It is given that the x-coordinate of point R is 4

$$\frac{8k+2}{k+1} = 4$$

$$8k+2 = 4k+4$$

$$4k = 2$$

$$k = \frac{1}{2}$$

Therefore, the coordinates of points R are

$$\left(4, \frac{-3}{\frac{1}{2}+1}, \frac{10\left(\frac{1}{2}\right)+4}{\frac{1}{2}+1}\right) = (4, -2, 6)$$

Question 6:

If A and B be the points (3, 4, 5) and (-1, 3, -7), respectively, find the equation of the set of points P such that $PA^2 + PB^2 = k^2$, where k is a constant.

Solution 6:

The coordinates of points A and B are given as (3, 4, 5) and (-1, 3, -7) respectively. Let the coordinates of point P be (x, y, z).
On using distance formula, we obtain

$$PA^2=(x-3)^2+(y-4)^2+(z-5)^2$$
$$=x^2+9-6x+y^2+16-8y+z^2+25-10z$$
$$=x^2-6x+y^2-8y+z^2-10z+50$$
$$PB^2=(x-1)^2+(y-3)^2+(z-7)^2$$
$$=x^2-2x+y^2-6y+z^2+14z+59$$
$$Now,\ if\ PA^2+PB^2=k^2,\ then$$
$$(x^2-6x+y^2-8y+z^2-10z+50)+(x^2+2x+y^2-6y+z2+14z+59)=k^2$$
$$\Rightarrow 2x^2+2y^2+2z^2-4x-14y+4z+109=k^2$$
$$\Rightarrow 2(x^2+y^2+z^2-2x-7y+2z)=k^2-109$$
$$\Rightarrow x^2+y^2+z^2-2x-7y+2z=\frac{k^2-109}{2}$$

Thus, the required equation is b $x^2+y^2+z^2-2x-7y+2z=\frac{k^2-109}{2}$.