

Chapter 8

Binomial Theorem

Exercise 8.1

Question 1:

Expand the expression $(1-2x)^5$

Solution 1:

By using Binomial Theorem, the expression $(1-2x)^5$ can be expanded as $(1-2x)^5$
 $= {}^5C_0(1)^5 - {}^5C_1(1)^4(2x) + {}^5C_2(1)^3(2x)^2 - {}^5C_3(1)^2(2x)^3 + {}^5C_4(1)^1(2x)^4 - {}^5C_5(2x)^5$
 $= 1 - 5(2x) + 10(4x)^2 - 10(8x^3) + 5(16x^4) - (32x^5)$
 $= 1 - 10x + 40x^2 - 80x^3 + 80x^4 - 32x^5$

Question 2:

Expand the expression $\left(\frac{2}{x} - \frac{x}{2}\right)^5$

Solution 2:

By using Binomial Theorem, the expression $\left(\frac{2}{x} - \frac{x}{2}\right)^5$ can be expanded as

$$\begin{aligned} \left(\frac{2}{x} - \frac{x}{2}\right)^5 &= {}^5C_0\left(\frac{2}{x}\right)^5 - {}^5C_1\left(\frac{2}{x}\right)^4\left(\frac{x}{2}\right) + {}^5C_2\left(\frac{2}{x}\right)^3\left(\frac{x}{2}\right)^2 - {}^5C_3\left(\frac{2}{x}\right)^2\left(\frac{x}{2}\right)^3 + {}^5C_4\left(\frac{2}{x}\right)\left(\frac{x}{2}\right)^4 - {}^5C_5\left(\frac{x}{2}\right)^5 \\ &= \frac{32}{x^5} - 5\left(\frac{16}{x^4}\right)\left(\frac{x}{2}\right) + 10\left(\frac{8}{x^3}\right)\left(\frac{x^2}{4}\right) - 10\left(\frac{4}{x^2}\right)\left(\frac{x^3}{8}\right) + 5\left(\frac{2}{x}\right)\left(\frac{x^4}{16}\right) - \frac{x^5}{32} \\ &= \frac{32}{x^5} - \frac{40}{x^3} + \frac{20}{x} - 5x + \frac{5}{8}x^3 - \frac{x^5}{32} \end{aligned}$$

Question 3:

Expand the expression $(2x-3)^6$

Solution 3:

By using Binomial Theorem, the expression $(2x-3)^6$ can be expanded as

$$\begin{aligned} (2x-3)^6 &= {}^6C_0(2x)^6 - {}^6C_1(2x)^5(3) + {}^6C_2(2x)^4(3)^2 - {}^6C_3(2x)^3(3)^3 + {}^6C_4(2x)^2(3)^4 - {}^6C_5(2x)(3)^5 + {}^6C_6(3)^6 \\ &= 64x^6 - 6(32x^5)(3) + 15(16x^4)(9) - 20(8x^3)(27) + 15(4x^2)(81) - 6(2x)(243) + 729 \\ &= 64x^6 - 576x^5 + 2160x^4 - 4320x^3 + 4860x^2 - 2916x + 729 \end{aligned}$$

Question 4:

Expand the expression $\left(\frac{x}{3} + \frac{1}{x}\right)^5$

Chapter 8

Binomial Theorem

Solution 4:

By using Binomial Theorem, the expression $\left(\frac{x}{3} + \frac{1}{x}\right)^5$ can be expanded as

$$\begin{aligned} \left(\frac{x}{3} + \frac{1}{x}\right)^5 &= {}^5C_0 \left(\frac{x}{3}\right)^5 + {}^5C_1 \left(\frac{x}{3}\right)^4 \left(\frac{1}{x}\right) + {}^5C_2 \left(\frac{x}{3}\right)^3 \left(\frac{1}{x}\right)^2 + {}^5C_3 \left(\frac{x}{3}\right)^2 \left(\frac{1}{x}\right)^3 + {}^5C_4 \left(\frac{x}{3}\right) \left(\frac{1}{x}\right)^4 + {}^5C_5 \left(\frac{1}{x}\right)^5 \\ &= \frac{x^5}{243} + 5 \left(\frac{x^4}{81}\right) \left(\frac{1}{x}\right) + 10 \left(\frac{x^3}{27}\right) \left(\frac{1}{x^2}\right) + 10 \left(\frac{x^2}{9}\right) \left(\frac{1}{x^3}\right) + 5 \left(\frac{x}{3}\right) \left(\frac{1}{x^4}\right) + \frac{1}{x^5} \\ &= \frac{x^5}{243} + \frac{5x^3}{81} + \frac{10x}{9x} + \frac{5}{3x^3} + \frac{1}{x^5} \end{aligned}$$

Question 5:

Expand $\left(x + \frac{1}{x}\right)^6$

Solution 5:

By using Binomial Theorem, the expression $\left(x + \frac{1}{x}\right)^6$ can be expanded as

$$\begin{aligned} \left(x + \frac{1}{x}\right)^6 &= {}^6C_0 (x)^6 + {}^6C_1 (x)^5 \left(\frac{1}{x}\right) + {}^6C_2 (x)^4 \left(\frac{1}{x}\right)^2 + {}^6C_3 (x)^3 \left(\frac{1}{x}\right)^3 + {}^6C_4 (x)^2 \left(\frac{1}{x}\right)^4 + {}^6C_5 (x) \left(\frac{1}{x}\right)^5 + {}^6C_6 \left(\frac{1}{x}\right)^6 \\ &= x^6 + 6(x)^5 \left(\frac{1}{x}\right) + 15(x)^4 \left(\frac{1}{x^2}\right) + 20(x)^3 \left(\frac{1}{x^3}\right) + 15(x)^2 \left(\frac{1}{x^4}\right) + 6(x) \left(\frac{1}{x^5}\right) + \frac{1}{x^6} \\ &= x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6} \end{aligned}$$

Question 6:

Using Binomial Theorem, evaluate $(96)^3$

Solution 6:

96 can be expressed as the sum or difference of two numbers whose powers are easier to calculate and then, binomial theorem can be applied.

It can be written that, $96 = 100 - 4$

$$\begin{aligned} \therefore (96)^3 &= (100 - 4)^3 \\ &= {}^3C_0 (100)^3 - {}^3C_1 (100)^2 (4) + {}^3C_2 (100)(4)^2 - {}^3C_3 (4)^3 \\ &= (100)^3 - 3(100)^2 (4) + 3(100)(4)^2 - (4)^3 \\ &= 1000000 - 120000 + 4800 - 64 \\ &= 884736 \end{aligned}$$

Question 7:

Using Binomial Theorem, evaluate $(102)^5$

Solution 7:

102 can be expressed as the sum or difference of two numbers whose powers are easier to calculate and then, binomial theorem can be applied.

It can be written that, $102 = 100 + 2$

$$\begin{aligned}\therefore (102)^5 &= (100 + 2)^5 \\ &= {}^5C_0(100)^5 + {}^5C_1(100)^4(2) + {}^5C_2(100)^3(2)^2 + {}^5C_3(100)^2(2)^3 + {}^5C_4(100)(2)^4 + {}^5C_5(2)^5 \\ &= 10000000000 + 1000000000 + 40000000 + 800000 + 8000 + 32 \\ &= 11040808032\end{aligned}$$

Question 8:

Using Binomial Theorem, evaluate $(101)^4$

Solution 8:

101 can be expressed as the sum or difference of two numbers whose powers are easier to calculate and then, binomial theorem can be applied.

It can be written that, $101 = 100 + 1$

$$\begin{aligned}\therefore (101)^4 &= (100 + 1)^4 \\ &= {}^4C_0(100)^4 + {}^4C_1(100)^3(1) + {}^4C_2(100)^2(1)^2 + {}^4C_3(100)(1)^3 + {}^4C_4(1)^4 \\ &= (100)^4 + 4(100)^3 + 6(100)^2 + 4(100) + (1)^4 \\ &= 100000000 + 4000000 + 60000 + 400 + 1 \\ &= 104060401\end{aligned}$$

Question 9:

Using Binomial Theorem, evaluate $(99)^5$

Solution 9:

99 can be written as the sum or difference of two numbers whose powers are easier to calculate and then, binomial theorem can be applied.

It can be written that, $99 = 100 - 1$

$$\begin{aligned}\therefore (99)^5 &= (100 - 1)^5 \\ &= {}^5C_0(100)^5 - {}^5C_1(100)^4(1) + {}^5C_2(100)^3(1)^2 - {}^5C_3(100)^2(1)^3 + {}^5C_4(100)(1)^4 - {}^5C_5(1)^5 \\ &= (100)^5 - 5(100)^4 + 10(100)^3 - 10(100)^2 + 5(100) - 1 \\ &= 10000000000 - 5000000000 + 100000000 - 100000 + 500 - 1 \\ &= 10010000500 - 500100001 \\ &= 9509900499\end{aligned}$$

Question 10:

Using Binomial Theorem, indicate which number is larger $(1.1)^{10000}$ or 1000.

Solution 10:

By splitting 1.1 and then applying Binomial Theorem, the first few terms of $(1.1)^{10000}$ be obtained as

$$\begin{aligned}(1.1)^{10000} &= (1 + 0.1)^{10000} \\ &= {}^{10000}C_0 + {}^{10000}C_1(1.1) + \text{Other positive terms} \\ &= 1 + 10000 \times 1.1 + \text{Other positive terms} \\ &= 1 + 11000 + \text{Other positive terms} \\ &> 1000\end{aligned}$$

Hence, $(1.1)^{10000} > 1000$.

Question 11:

Find $(a+b)^4 - (a-b)^4$. Hence, evaluate. $(\sqrt{3} + \sqrt{2})^4 - (\sqrt{3} - \sqrt{2})^4$

Solution 11:

Using Binomial Theorem, the expressions, $(a+b)^4$ and $(a-b)^4$, can be expanded as

$$\begin{aligned}(a+b)^4 &= {}^4C_0a^4 + {}^4C_1a^3b + {}^4C_2a^2b^2 + {}^4C_3ab^3 + {}^4C_4b^4 \\ (a-b)^4 &= {}^4C_0a^4 - {}^4C_1a^3b + {}^4C_2a^2b^2 - {}^4C_3ab^3 + {}^4C_4b^4 \\ \therefore (a+b)^4 - (a-b)^4 &= {}^4C_0a^4 + {}^4C_1a^3b + {}^4C_2a^2b^2 + {}^4C_3ab^3 + {}^4C_4b^4 - [{}^4C_0a^4 - {}^4C_1a^3b + {}^4C_2a^2b^2 - {}^4C_3ab^3 + {}^4C_4b^4] \\ &= 2({}^4C_1a^3b + {}^4C_3ab^3) = 2(4a^3b + 4ab^3) \\ &= 8ab(a^2 + b^2)\end{aligned}$$

By putting $a = \sqrt{3}$ and $b = \sqrt{2}$, we obtain

$$\begin{aligned}(\sqrt{3} + \sqrt{2})^4 - (\sqrt{3} - \sqrt{2})^4 &= 8(\sqrt{3})(\sqrt{2})\{(\sqrt{3})^2 + (\sqrt{2})^2\} \\ &= 8(\sqrt{6})\{3 + 2\} = 40\sqrt{6}\end{aligned}$$

Question 12:

Find $(x+1)^6 + (x-1)^6$. Hence or otherwise evaluate. $(\sqrt{2} + 1)^6 + (\sqrt{2} - 1)^6$

Solution 12:

Using Binomial Theorem, the expression, $(x+1)^6$ and $(x-1)^6$, can be expanded as

Chapter 8

Binomial Theorem

$$(x+1)^6 = {}^6C_0x^6 + {}^6C_1x^5 + {}^6C_2x^4 + {}^6C_3x^3 + {}^6C_4x^2 + {}^6C_5x + {}^6C_6$$

$$(x-1)^6 = {}^6C_0x^6 - {}^6C_1x^5 + {}^6C_2x^4 - {}^6C_3x^3 + {}^6C_4x^2 - {}^6C_5x + {}^6C_6$$

$$\therefore (x+1)^6 + (x-1)^6 = 2 \left[{}^6C_0x^6 + {}^6C_2x^4 + {}^6C_4x^2 + {}^6C_6 \right]$$

$$= 2 \left[x^6 + 15x^4 + 15x^2 + 1 \right]$$

By putting $x = \sqrt{2}$ we obtain

$$\left(\sqrt{2}+1\right)^6 + \left(\sqrt{2}-1\right)^6 = 2 \left[\left(\sqrt{2}\right)^6 + 15\left(\sqrt{2}\right)^4 + 15\left(\sqrt{2}\right)^2 + 1 \right]$$

$$= 2(8 + 15 \times 4 + 15 \times 2 + 1)$$

$$= 2(8 + 60 + 30 + 1)$$

$$= 2(99) = 198$$

Question 13:

Show that $9^{n+1} - 8n - 9$ is divisible by 64, whenever n is a positive integer.

Solution 13:

In order to show that $9^{n+1} - 8n - 9$ is divisible by 64, it has to be prove that, $9^{n+1} - 8n - 9 = 64k$, where k is some natural number

By Binomial Theorem,

$$(1+a)^m = {}^mC_0 + {}^mC_1a + {}^mC_2a^2 + \dots + {}^mC_ma^m$$

For $a=8$ and $m=n+1$, we obtain

$$(1+8)^{n+1} = {}^{n+1}C_0 + {}^{n+1}C_1(8) + {}^{n+1}C_2(8)^2 + \dots + {}^{n+1}C_{n+1}(8)^{n+1}$$

$$\Rightarrow 9^{n+1} = 1 + (n+1)(8) + 8^2 \left[{}^{n+1}C_2 + {}^{n+1}C_3 \times 8 + \dots + {}^{n+1}C_{n+1}(8)^{n-1} \right]$$

$$\Rightarrow 9^{n+1} = 9 + 8n + 64 \left[{}^{n+1}C_2 + {}^{n+1}C_3 \times 8 + \dots + {}^{n+1}C_{n+1}(8)^{n-1} \right]$$

$$\Rightarrow 9^{n+1} - 8n - 9 = 64k, \text{ where } k = {}^{n+1}C_2 + {}^{n+1}C_3 \times 8 + \dots + {}^{n+1}C_{n+1}(8)^{n-1} \text{ is a natural number}$$

Thus, $9^{n+1} - 8n - 9$ is divisible by 64, whenever n is a positive integer.

Question 14:

Prove that $\sum_{r=0}^n 3^r {}^nC_r = 4^n$

Solution 14:

By Binomial Theorem,

$$\sum_{r=0}^n {}^nC_r a^{n-r} b^r = (a+b)^n$$

By putting $b=3$ and $a=1$ in the above equation, we obtain

$$\sum_{r=0}^n {}^nC_r (1)^{n-r} (3)^r = (1+3)^n$$

$$\Rightarrow \sum_{r=0}^n 3^r {}^nC_r = 4^n$$

Hence proved.

Exercise 8.2

Question 1:

Find the coefficient of x^5 in $(x+3)^8$

Solution 1:

It is known that $(r+1)^{th}$ term, (T_{r+1}) , in the binomial expansion of $(a+b)^n$ is given by

$$T_{r+1} = {}^nC_r a^{n-r} b^r$$

Assuming that x^5 occurs in the $(r+1)^{th}$ term of the expansion $(x+3)^8$, we obtain

$$T_{r+1} = {}^8C_r (x)^{8-r} (3)^r$$

Comparing the indices of x in x^5 in T_{r+1} ,

We obtain $r = 3$

Thus, the coefficient of x^5 is ${}^8C_3 (3)^3 = \frac{8!}{3!5!} \times 3^3 = \frac{8 \cdot 7 \cdot 6 \cdot 5!}{3 \cdot 2 \cdot 5!} \cdot 3^3 = 1512$.

Question 2:

Find the coefficient of a^5b^7 in $(a-2b)^{12}$

Solution 2:

It is known that $(r+1)^{th}$ term, (T_{r+1}) , in the binomial expansion of $(a+b)^n$ is given by

$$T_{r+1} = {}^nC_r a^{n-r} b^r$$

Assuming that a^5b^7 occurs in the $(r+1)^{th}$ term of the expansion $(a-2b)^{12}$, we obtain

$$T_{r+1} = {}^{12}C_r (a)^{12-r} (-2b)^r = {}^{12}C_r (-2)^r (a)^{12-r} (b)^r$$

Comparing the indices of a and b in a^5b^7 in T_{r+1} ,

We obtain $r = 7$

Thus, the coefficient of a^5b^7 is

$${}^{12}C_7 (-2)^7 = \frac{12!}{7!5!} \cdot 2^7 = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7!}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 7!} \cdot (-2)^7 = -(792)(128) = -101376.$$

Chapter 8

Binomial Theorem

Question 3:

Write the general term in the expansion of $(x^2 - y)^6$

Solution 3:

It is known that the general term T_{r+1} {which is the $(r+1)^{\text{th}}$ term} in the binomial expansion of $(a+b)^n$ is given by $T_{r+1} = {}^nC_r a^{n-r} b^r$.

Thus, the general term in the expansion of $(x^2 - y^6)$ is

$$T_{r+1} = {}^6C_r (x^2)^{6-r} (-y)^r = (-1)^r {}^6C_r \cdot x^{12-2r} \cdot y^r$$

Question 4:

Write the general term in the expansion of $(x^2 - yx)^{12}$, $x \neq 0$

Solution 4:

It is known that the general term T_{r+1} {which is the $(r+1)^{\text{th}}$ term} in the binomial expansion of $(a+b)^n$ is given by $T_{r+1} = {}^nC_r a^{n-r} b^r$.

Thus, the general term in the expansion of $(x^2 - yx)^{12}$ is

$$T_{r+1} = {}^{12}C_r (x^2)^{12-r} (-yx)^r = (-1)^r {}^{12}C_r \cdot x^{24-2r} \cdot y^r = (-1)^r {}^{12}C_r \cdot x^{24-2r} \cdot y^r$$

Question 5:

Find the 4th term in the expansion of $(x - 2y)^{12}$.

Solution 5:

It is known $(r+1)^{\text{th}}$ term, T_{r+1} , in the binomial expansion of $(a+b)^n$ is given by $T_{r+1} = {}^nC_r a^{n-r} b^r$.

Thus, the 4th term in the expansion of $(x^2 - 2y)^{12}$ is

$$T_4 = T_{3+1} = {}^{12}C_3 (x)^{12-3} (-2y)^3 = (-1)^3 \cdot \frac{12!}{3!9!} \cdot x^9 \cdot (2)^3 \cdot y^3 = -\frac{12 \cdot 11 \cdot 10}{3 \cdot 2} \cdot (2)^3 x^9 y^3 = -1760 x^9 y^3$$

Question 6:

Find the 13th term in the expansion of $\left(9x - \frac{1}{3\sqrt{x}}\right)^{18}$, $x \neq 0$

Solution 6:

Chapter 8

Binomial Theorem

It is known $(r+1)^{th}$ term, T_{r+1} , in the binomial expansion of $(a+b)^n$ is given by $T_{r+1} = {}^nC_r a^{n-r} b^r$.

Thus, the 13th term in the expansion of $\left(9x - \frac{1}{3\sqrt{x}}\right)^{18}$ is

$$\begin{aligned} T_{13} &= T_{12+1} = {}^{18}C_{12} (9x)^{18-12} \left(-\frac{1}{3\sqrt{x}}\right)^{12} \\ &= (-1)^{12} \frac{18!}{12!6!} (9)^6 (x)^6 \left(\frac{1}{3}\right)^{12} \left(\frac{1}{\sqrt{x}}\right)^{12} \\ &= \frac{18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12!}{12! \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} \cdot x^6 \left(\frac{1}{x^6}\right) \cdot 3^{12} \left(\frac{1}{3^{12}}\right) \quad \left[9^6 = (3^2)^6 = 3^{12}\right] \\ &= 18564 \end{aligned}$$

Question 7:

Find the middle terms in the expansions of $\left(3 - \frac{x^3}{6}\right)^7$

Solution 7:

It is known that in the expansion of $(a+b)^n$, in n is odd, then there are two middle terms,

Namely $\left(\frac{n+1}{2}\right)^{th}$ term and $\left(\frac{n+1}{2} + 1\right)^{th}$ term.

Therefore, the middle terms in the expansion $\left(3 - \frac{x^3}{6}\right)^7$ are $\left(\frac{7+1}{2}\right)^{th} = 4^{th}$ and $\left(\frac{7+1}{2} + 1\right)^{th} = 5^{th}$ term

$$\begin{aligned} T_4 &= T_{3+1} = {}^7C_3 (3)^{7-3} \left(-\frac{x^3}{6}\right)^3 = (-1)^3 \frac{7!}{3!4!} \cdot 3^4 \cdot \frac{x^9}{6^3} \\ &= -\frac{7 \cdot 6 \cdot 5 \cdot 4!}{3 \cdot 2 \cdot 4!} \cdot 3^4 \cdot \frac{1}{2^3 \cdot 3^3} \cdot x^9 = -\frac{105}{8} x^9 \\ T_5 &= T_{4+1} = {}^7C_4 (3)^{7-4} \left(-\frac{x^3}{6}\right)^4 = (-1)^4 \frac{7!}{4!3!} \cdot 3^3 \cdot \frac{x^{12}}{6^4} \\ &= \frac{7 \cdot 6 \cdot 5 \cdot 4!}{4! \cdot 3 \cdot 2} \cdot \frac{3^3}{2^4 \cdot 3^4} \cdot x^{12} = \frac{35}{48} x^{12} \end{aligned}$$

Thus, the middle terms in the expansion of $\left(3 - \frac{x^3}{6}\right)^7$ are $-\frac{105}{8} x^9$ and $\frac{35}{48} x^{12}$.

Question 8:

Find the middle terms in the expansion of $\left(\frac{x}{3} + 9y\right)^{10}$

Solution 8:

It is known that in the expansion of $(a+b)^n$, in n is even, then the middle term is

$\left(\frac{n}{2} + 1\right)^{th}$ term.

Therefore, the middle term in the expansion of $\left(\frac{x}{3} + 9y\right)^{10}$ is $\left(\frac{10}{2} + 1\right)^{th} = 6^{th}$

$$\begin{aligned} T_4 = T_{5+1} &= {}^{10}C_5 \left(\frac{x}{3}\right)^{10-5} (9y)^5 = \frac{10!}{5!5!} \cdot \frac{x^5}{3^5} \cdot 9^5 \cdot y^5 \\ &= \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5!}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 5!} \cdot \frac{1}{3^5} \cdot 3^{10} \cdot x^5 y^5 \quad \left[9^5 = (3^2)^5 = 3^{10}\right] \\ &= 252 \times 3^5 \cdot x^5 \cdot y^5 = 6123 x^5 y^5 \end{aligned}$$

Thus, the middle term in the expansion of $\left(\frac{x}{3} + 9y\right)^{10}$ is $6123 x^5 y^5$.

Question 9:

In the expansion of $(1+a)^{m+n}$, prove that coefficients of a^m and a^n are equal.

Solution 9:

It is known that $(r+1)^{th}$ term, (T_{r+1}) , in the binomial expansion of $(a+b)^n$ is given by

$$T_{r+1} = {}^nC_r a^{n-r} b^r$$

Assuming that a^m occurs in the $(r+1)^{th}$ term of the expansion $(1+a)^{m+n}$, we obtain

$$T_{r+1} = {}^{m+n}C_r (1)^{m+n-r} (a)^r = {}^{m+n}C_r a^r$$

Comparing the indices of a in a^m in T_{r+1} ,

We obtain $r = m$

Therefore, the coefficient of a^m is

$${}^{m+n}C_m = \frac{(m+n)!}{m!(m+n-m)!} = \frac{(m+n)!}{m!n!} \dots\dots\dots (1)$$

Assuming that a^n occurs in the $(k+1)^{th}$ term of the expansion $(1+a)^{m+n}$, we obtain

$$T_{k+1} = {}^{m+n}C_k (1)^{m+n-k} (a)^k = {}^{m+n}C_k (a)^k$$

Comparing the indices of a in a^n and in T_{k+1} ,

We obtain

$$k = n$$

Therefore, the coefficient of a^n is

$${}^{m+n}C_n = \frac{(m+n)!}{n!(m+n-n)!} = \frac{(m+n)!}{n!m!} \dots\dots (2)$$

Thus, from (1) and (2), it can be observed that the coefficients of a^m and a^n in the expansion of $(1+a)^{m+n}$ are equal.

Question 10:

The coefficients of the $(r-1)^{th}$, r^{th} and $(r+1)^{th}$ terms in the expansion of $(x+1)^n$ are in the ratio 1:3:5. Find n and r .

Solution 10:

It is known that $(k+1)^{th}$ term, (T_{k+1}) , in the binomial expansion of $(a+b)^n$ is given by

$$T_{k+1} = {}^nC_k a^{n-k} b^k.$$

Therefore, $(r-1)^{th}$ term in the expansion of $(x+1)^n$ is

$$T_{r-1} = {}^nC_{r-2} (x)^{n-(r-2)} (1)^{(r-2)} = {}^nC_{r-2} x^{n-r+2}$$

$(r+1)$ term in the expansion of $(x+1)^n$ is

$$T_{r+1} = {}^nC_r (x)^{n-r} (1)^r = {}^nC_r x^{n-r}$$

r^{th} term in the expansion of $(x+1)^n$ is

$$T_r = {}^nC_{r-1} (x)^{n-(r-1)} (1)^{(r-1)} = {}^nC_{r-1} x^{n-r+1}$$

Therefore, the coefficients of the $(r-1)^{th}$, r^{th} and $(r+1)^{th}$ terms in the expansion of $(x+1)^n$

${}^nC_{r-2}$, ${}^nC_{r-1}$, and nC_r are respectively. Since these coefficients are in the ratio 1:3:5, we obtain

$$\frac{{}^nC_{r-2}}{{}^nC_{r-1}} = \frac{1}{3} \text{ and } \frac{{}^nC_{r-1}}{{}^nC_r} = \frac{3}{5}$$

$$\frac{{}^nC_{r-2}}{{}^nC_{r-1}} = \frac{n!}{(r-2)!(n-r+2)!} \times \frac{(r-1)!(n-r+1)!}{n!} = \frac{(r-1)(r-2)!(n-r+1)!}{(r-2)!(n-r+2)!(n-r+1)!}$$

$$= \frac{r-1}{n-r+2}$$

$$\therefore \frac{r-1}{n-r+2} = \frac{1}{3}$$

$$\Rightarrow 3r-3 = n-r+2$$

$$\Rightarrow n-4r+5 = 0 \dots\dots (1)$$

$$\frac{{}^nC_{r-1}}{{}^nC_r} = \frac{n!}{(r-1)!(n-r+1)!} \times \frac{r!(n-r)!}{n!} = \frac{r(r-1)!(n-r)!}{(r-1)!(n-r+1)(n-r)!}$$

$$= \frac{r}{n-r+1}$$

$$\therefore \frac{r}{n-r+1} = \frac{3}{5}$$

$$\Rightarrow 5r = 3n - 3r + 3$$

$$\Rightarrow 3n - 8r + 3 = 0 \quad \dots\dots(2)$$

Multiplying (1) by 3 and subtracting it from (2), we obtain

$$4r - 12 = 0$$

$$\Rightarrow r = 3$$

Putting the value of r in (1), we obtain n

$$-12 + 5 = 0$$

$$\Rightarrow n = 7$$

Thus, $n = 7$ and $r = 3$

Question 11:

Prove that the coefficient of x^n in the expansion of $(1+x)^{2n}$ is twice the coefficient of x^n in the expansion of $(1+x)^{2n-1}$.

Solution 11:

It is known that $(r+1)^{th}$ term, (T_{r+1}) , in the binomial expansion of $(a+b)^n$ is given by

$$T_{r+1} = {}^nC_r a^{n-r} b^r.$$

Assuming that x^n occurs in the $(r+1)^{th}$ term of the expansion of $(1+x)^{2n}$, we obtain

$$T_{r+1} = {}^{2n}C_r (1)^{2n-r} (x)^r = {}^{2n}C_r (x)^r$$

Comparing the indices of x in x^n and in T_{r+1} , we obtain $r = n$

Therefore, the coefficient of x^n in the expansion of $(1+x)^{2n}$ is

$${}^{2n}C_n = \frac{(2n)!}{n!(2n-n)!} = \frac{(2n)!}{n!n!} = \frac{(2n)!}{(n!)^2} \dots\dots(1)$$

Assuming that x^n occurs in the $(k+1)^{th}$ term of the expansion of $(1+x)^{2n-1}$, we obtain

$$T_{k+1} = {}^{2n-1}C_k (1)^{2n-1-k} (x)^k = {}^{2n-1}C_k (x)^k$$

Comparing the indices of x in x^n and in T_{k+1} , we obtain $k = n$

Therefore, the coefficient of x^n in the expansion of $(1+x)^{2n-1}$ is

$$\begin{aligned} {}^{2n-1}C_n &= \frac{(2n-1)!}{n!(2n-1-n)!} = \frac{(2n-1)!}{n!(n-1)!} \\ &= \frac{2n \cdot (2n-1)!}{2n \cdot n!(n-1)!} = \frac{(2n)!}{2 \cdot n!n!} = \frac{1}{2} \left[\frac{(2n)!}{(n!)^2} \right] \dots\dots(2) \end{aligned}$$

From (1) and (2), it is observed that

$$\frac{1}{2} ({}^{2n}C_n) = {}^{2n-1}C_n$$

$$\Rightarrow {}^{2n}C_n = 2({}^{2n-1}C_n)$$

Therefore, the coefficient of x^n expansion of $(1+x)^{2n}$ is twice the coefficient of x^n in the expansion of $(1+x)^{2n-1}$.

Hence proved.

Question 12:

Find a positive value of m for which the coefficient of x^2 in the expansion $(1+x)^m$ is 6.

Solution 12:

It is known that $(r+1)^{th}$ term, (T_{r+1}) , in the binomial expansion of $(a+b)^n$ is given by

$$T_{r+1} = {}^nC_r a^{n-r} b^r.$$

Assuming that x^2 occurs in the $(r+1)^{th}$ term of the expansion of $(1+x)^m$, we obtain

$$T_{r+1} = {}^mC_r (1)^{m-r} (x)^r = {}^mC_r (x)^r$$

Comparing the indices of x in x^2 and in T_{r+1} , we obtain $r = 2$

Therefore, the coefficient of x^2 is mC_2

It is given that the coefficient of x^2 in the expansion $(1+x)^m$ is 6.

$$\therefore {}^mC_2 = 6$$

$$\Rightarrow \frac{m!}{2!(m-2)!} = 6$$

$$\Rightarrow \frac{m(m-1)(m-2)!}{2 \times (m-2)!} = 6$$

$$\Rightarrow m(m-1) = 12$$

$$\Rightarrow m^2 - m - 12 = 0$$

$$\Rightarrow m^2 - 4m + 3m - 12 = 0$$

$$\Rightarrow m(m-4) + 3(m-4) = 0$$

$$\Rightarrow (m-4)(m+3) = 0$$

$$\Rightarrow (m-4) = 0 \text{ or } (m+3) = 0$$

$$\Rightarrow m = 4 \text{ or } m = -3$$

Thus, the positive value of m , for which the coefficient of x^2 in the expansion $(1+x)^m$ is 6, is 4.

Miscellaneous Exercise

Question 1:

Find a , b and n in the expansion of $(a+b)^n$ if the first three terms of the expansion are 729, 7290 and 30375, respectively.

Solution 1:

It is known that $(r+1)^{\text{th}}$ term, (T_{r+1}) , in the binomial expansion of $(a+b)^n$ is given by

$$T_{r+1} = {}^nC_r a^{n-r} b^r.$$

The first three terms of the expansion are given as 729, 7290 and 30375 respectively.

Therefore, we obtain

$$T_1 = {}^nC_0 a^{n-0} b^0 = a^n = 729 \dots\dots(1)$$

$$T_2 = {}^nC_1 a^{n-1} b^1 = na^{n-1}b = 7290 \dots\dots(2)$$

$$T_3 = {}^nC_2 a^{n-2} b^2 = \frac{n(n-1)}{2} a^{n-2} b^2 = 30375 \dots\dots(3)$$

Dividing (2) by (1), we obtain

$$\frac{na^{n-1}b}{a^n} = \frac{7290}{729}$$

$$\Rightarrow \frac{nb}{a} = 10 \dots\dots(4)$$

Dividing (3) by (2), we obtain

$$\frac{n(n-1)a^{n-2}b^2}{2na^{n-1}b} = \frac{30375}{7290}$$

$$\Rightarrow \frac{(n-1)b}{2a} = \frac{30375}{7290}$$

$$\Rightarrow \frac{(n-1)b}{a} = \frac{30375 \times 2}{7290} = \frac{25}{3}$$

$$\Rightarrow \frac{nb}{a} - \frac{b}{a} = \frac{25}{3}$$

$$\Rightarrow 10 - \frac{b}{a} = \frac{25}{3} \quad [\text{Using (4)}]$$

$$\Rightarrow \frac{b}{a} = 10 - \frac{25}{3} = \frac{5}{3} \dots\dots(5)$$

From (4) and (5), we obtain

$$n \cdot \frac{5}{3} = 10$$

$$\Rightarrow n = 6$$

Substituting $n = 6$ in equation (1), we obtain a^6

$$= 729$$

$$\Rightarrow a = \sqrt[6]{729} = 3$$

From (5), we obtain

$$\frac{b}{3} = \frac{5}{3} \Rightarrow b = 5$$

Thus, $a = 3$, $b = 5$, and $n = 6$.

Question 2:

Find a if the coefficients of x^2 and x^3 in the expansion of $(3+ax)^9$ are equal.

Solution 2:

It is known that $(r+1)^{th}$ term, (T_{r+1}) , in the binomial expansion of $(a+b)^n$ is given by

$$T_{r+1} = {}^nC_r a^{n-r} b^r.$$

Assuming that x^2 occurs in the $(r+1)^{th}$ term in the expansion of $(3+ax)^9$, we obtain

$$T_{r+1} = {}^9C_r (3)^{9-r} (ax)^r = {}^9C_r (3)^{9-r} a^r x^r$$

Comparing the indices of x in x^2 and in T_{r+1} , we obtain

$$r = 2$$

Thus, the coefficient of x^2 is

$${}^9C_2 (3)^{9-2} a^2 = \frac{9!}{2!7!} (3)^7 a^2 = 36(3)^7 a^2$$

Assuming that x^3 occurs in the $(k+1)^{th}$ term in the expansion of $(3+ax)^9$, we obtain

$$T_{k+1} = {}^9C_k (3)^{9-k} (ax)^k = {}^9C_k (3)^{9-k} a^k x^k$$

Comparing the indices of x in x^3 and in T_{k+1} , we obtain $k = 3$

Thus, the coefficient of x^3 is

$${}^9C_3 (3)^{9-3} a^3 = \frac{9!}{3!6!} (3)^6 a^3 = 84(3)^6 a^3$$

It is given that the coefficient of x^2 and x^3 are the same.

$$84(3)^6 a^3 = 36(3)^7 a^2$$

$$\Rightarrow 84a = 36 \times 3$$

$$\Rightarrow a = \frac{36 \times 3}{84} = \frac{104}{84}$$

$$\Rightarrow a = \frac{9}{7}$$

Thus, the required value of a is $9/7$.

Question 3:

Find the coefficient of x^5 in the product $(1+2x)^6 (1-x)^7$ using binomial theorem.

Solution 3:

Chapter 8

Binomial Theorem

Using Binomial Theorem, the expressions, $(1+2x)^6$ and $(1-x)^7$, can be expanded as

$$\begin{aligned}(1+2x)^6 &= {}^6C_0 + {}^6C_1(2x) + {}^6C_2(2x)^2 + {}^6C_3(2x)^3 + {}^6C_4(2x)^4 + {}^6C_5(2x)^5 + {}^6C_6(2x)^6 \\&= 1 + 6(2x) + 15(2x)^2 + 20(2x)^3 + 15(2x)^4 + 6(2x)^5 + (2x)^6 \\&= 1 + 12x + 60x^2 + 160x^3 + 240x^4 + 192x^5 + 64x^6 \\(1-x)^7 &= {}^7C_0 - {}^7C_1(x) + {}^7C_2(x)^2 - {}^7C_3(x)^3 + {}^7C_4(x)^4 - {}^7C_5(x)^5 + {}^7C_6(x)^6 - {}^7C_7(x)^7 \\&= 1 - 7x + 21x^2 - 35x^3 + 35x^4 - 21x^5 + 7x^6 - x^7 \\&\therefore (1+2x)^6(1-x)^7 \\&= (1+12x+60x^2+160x^3+240x^4+192x^5+64x^6)(1-7x+21x^2-35x^3+35x^4-21x^5+7x^6-x^7)\end{aligned}$$

The complete multiplication of the two brackets is not required to be carried out. Only those terms, which involve x^5 , are required.

The terms containing x^5 are

$$\begin{aligned}1(-21x^5) + (12x)(35x^4) + (60x^2)(-35x^3) + (160x^3)(21x^2) + (240x^4)(-7x) + (192x^5)(1) \\= 171x^5\end{aligned}$$

Thus, the coefficient of x^5 in the given product is 171.

Question 4:

If a and b are distinct integers, prove that $a-b$ is a factor of $a^n - b^n$, whenever n is a positive integer. [**Hint:** write $a^n = (a-b+b)^n$ and expand]

Solution 4:

In order to prove that $(a-b)$ is a factor of $(a^n - b^n)$, it has to be proved that

$a^n - b^n = k(a-b)$, where k is some natural number

It can be written that, $a = a-b+b$

$$\begin{aligned}\therefore a^n &= (a-b+b)^n = [(a-b)+b]^n \\&= {}^nC_0(a-b)^n + {}^nC_1(a-b)^{n-1}b + \dots + {}^nC_{n-1}(a-b)b^{n-1} + {}^nC_nb^n \\&= (a-b)^n + {}^nC_1(a-b)^{n-1}b + \dots + {}^nC_{n-1}(a-b)b^{n-1} + b^n \\&\Rightarrow a^n - b^n = (a-b) \left[(a-b)^{n-1} + {}^nC_1(a-b)^{n-2}b + \dots + {}^nC_{n-1}b^{n-1} \right] \\&\Rightarrow a^n - b^n = k(a-b)\end{aligned}$$

Where, $k = \left[(a-b)^{n-1} + {}^nC_1(a-b)^{n-2}b + \dots + {}^nC_{n-1}b^{n-1} \right]$ is a natural number

This shows that $(a-b)$ is a factor of $(a^n - b^n)$, where n is a positive integer.

Question 5:

Evaluate $(\sqrt{3} + \sqrt{2})^6 - (\sqrt{3} - \sqrt{2})^6$

Solution 5:

Firstly, the expression $(a + b)^6 - (a - b)^6$ is simplified by using Binomial Theorem. This can be done as

$$(a + b)^6 = {}^6C_0a^6 + {}^6C_1a^5b + {}^6C_2a^4b^2 + {}^6C_3a^3b^3 + {}^6C_4a^2b^4 + {}^6C_5a^1b^5 + {}^6C_6b^6$$

$$= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

$$(a - b)^6 = {}^6C_0a^6 - {}^6C_1a^5b + {}^6C_2a^4b^2 - {}^6C_3a^3b^3 + {}^6C_4a^2b^4 - {}^6C_5a^1b^5 + {}^6C_6b^6$$

$$= a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6$$

$$\therefore (a + b)^6 - (a - b)^6 = 2[6a^5b + 20a^3b^3 + 6ab^5]$$

Putting $a = \sqrt{3}$ and $b = \sqrt{2}$, we obtain

$$(\sqrt{3} + \sqrt{2})^6 - (\sqrt{3} - \sqrt{2})^6 = 2\left[6(\sqrt{3})^5(\sqrt{2}) + 20(\sqrt{3})^3(\sqrt{2})^3 + 6(\sqrt{3})(\sqrt{2})^5\right]$$

$$= 2[54\sqrt{6} + 120\sqrt{6} + 24\sqrt{6}]$$

$$= 2 \times 198\sqrt{6}$$

$$= 396\sqrt{6}$$

Question 6:

Find the value of $(a^2 + \sqrt{a^2 - 1})^4 + (a^2 - \sqrt{a^2 - 1})^4$

Solution 6:

Firstly, the expression $(x + y)^4 + (x - y)^4$ is simplified by using Binomial Theorem.

This can be done as

$$(x + y)^4 = {}^4C_0x^4 + {}^4C_1x^3y + {}^4C_2x^2y^2 + {}^4C_3xy^3 + {}^4C_4y^4$$

$$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

$$(x - y)^4 = {}^4C_0x^4 - {}^4C_1x^3y + {}^4C_2x^2y^2 - {}^4C_3xy^3 + {}^4C_4y^4$$

$$= x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$$

$$\therefore (x + y)^4 + (x - y)^4 = 2(x^4 + 6x^2y^2 + y^4)$$

Putting $x = a^2$ and $y = \sqrt{a^2 - 1}$, we obtain

$$(a^2 + \sqrt{a^2 - 1})^4 + (a^2 - \sqrt{a^2 - 1})^4 = 2\left[(a^2)^4 + 6(a^2)^2(\sqrt{a^2 - 1})^2 + (\sqrt{a^2 - 1})^4\right]$$

$$= 2\left[a^8 + 6a^4(a^2 - 1) + (a^2 - 1)^2\right]$$

$$= 2[a^8 + 6a^6 - 6a^4 + a^4 - 2a^2 + 1]$$

$$= 2[a^8 + 6a^6 - 5a^4 - 2a^2 + 1]$$

$$= 2a^8 + 12a^6 - 10a^4 - 4a^2 + 2$$

Question 7:

Find an approximation of $(0.99)^5$ using the first three terms of its expansion.

Solution 7:

$$0.99 = 1 - 0.01$$

$$\therefore (0.99)^5 = (1 - 0.01)^5$$

$$= {}^5C_0(1)^5 - {}^5C_1(1)^4(0.01) + {}^5C_2(1)^3(0.01)^2 \quad [\text{Approximately}]$$

$$= 1 - 5(0.01) + 10(0.01)^2$$

$$= 1 - 0.05 + 0.001$$

$$= 1.001 - 0.05$$

$$= 0.951$$

Thus, the value of $(0.99)^5$ is approximately 0.951.

Question 8:

Find n , if the ratio of the fifth term from the beginning to the fifth term from the end in the expansion of $\left(\sqrt[4]{2} + \frac{1}{\sqrt[4]{3}}\right)^n$ is $\sqrt{6}:1$.

Solution 8:

In the expansion, $(a + b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + \dots + {}^nC_{n-1}ab^{n-1} + {}^nC_nb^n$

Fifth term from the beginning $= {}^nC_4a^{n-4}b^4$

Fifth term from the end $= {}^nC_4a^4b^{n-4}$

Therefore, it is evident that in the expansion of $\left(\sqrt[4]{2} + \frac{1}{\sqrt[4]{3}}\right)^n$ are fifth term from the beginning

is ${}^nC_4\left(\sqrt[4]{2}\right)^{n-4}\left(\frac{1}{\sqrt[4]{3}}\right)^4$ and the fifth term from the end is ${}^nC_{n-4}\left(\sqrt[4]{2}\right)^4\left(\frac{1}{\sqrt[4]{3}}\right)^{n-4}$

$${}^nC_4\left(\sqrt[4]{2}\right)^{n-4}\left(\frac{1}{\sqrt[4]{3}}\right)^4 = {}^nC_4\frac{\left(\sqrt[4]{2}\right)^n}{\left(\sqrt[4]{2}\right)^4} \cdot \frac{1}{3} = \frac{n!}{6 \cdot 4!(n-4)!}\left(\sqrt[4]{2}\right)^n \dots (1)$$

$${}^nC_{n-4}\left(\sqrt[4]{2}\right)^4\left(\frac{1}{\sqrt[4]{3}}\right)^{n-4} = {}^nC_{n-4}\frac{\left(\sqrt[4]{3}\right)^4}{\left(\sqrt[4]{3}\right)^n} = {}^nC_{n-4} \cdot 2 \cdot \frac{3}{\left(\sqrt[4]{3}\right)^n} = \frac{6n!}{(n-4)!4!} \cdot \frac{1}{\left(\sqrt[4]{3}\right)^n} \dots (2)$$

Chapter 8

Binomial Theorem

It is given that the ratio of the fifth term from the beginning to the fifth term from the end is $\sqrt{6}:1$. Therefore, from (1) and (2), we obtain

$$\frac{n!}{6 \cdot 4!(n-4)!} (\sqrt[4]{2})^n : \frac{6n!}{(n-4)!4!} \cdot \frac{1}{(\sqrt[4]{3})^n} = \sqrt{6}:1$$

$$\Rightarrow \frac{(\sqrt[4]{2})^n}{6} : \frac{6}{(\sqrt[4]{3})^n} = \sqrt{6}:1$$

$$\Rightarrow \frac{(\sqrt[4]{2})^n}{6} \times \frac{(\sqrt[4]{3})^n}{6} = \sqrt{6}$$

$$\Rightarrow (\sqrt[4]{6})^n = 36\sqrt{6}$$

$$\Rightarrow 6^{n/4} = 6^{5/2}$$

$$\Rightarrow \frac{n}{4} = \frac{5}{2}$$

$$\Rightarrow n = 4 \times \frac{5}{2} = 10$$

Thus, the value of n is 10.

Question 9:

Expand using Binomial Theorem $\left(1 + \frac{x}{2} - \frac{2}{x}\right)^4$, $x \neq 0$

Solution 9:

$$\begin{aligned} & \left(1 + \frac{x}{2} - \frac{2}{x}\right)^4 \\ &= {}^nC_0 \left(1 + \frac{x}{2}\right)^4 - {}^nC_1 \left(1 + \frac{x}{2}\right)^3 \left(\frac{2}{x}\right) + {}^nC_2 \left(1 + \frac{x}{2}\right)^2 \left(\frac{2}{x}\right)^2 - {}^nC_3 \left(1 + \frac{x}{2}\right) \left(\frac{2}{x}\right)^3 + {}^nC_4 \left(\frac{2}{x}\right)^4 \\ &= \left(1 + \frac{x}{2}\right)^4 - 4 \left(1 + \frac{x}{2}\right)^3 \left(\frac{2}{x}\right) + 6 \left(1 + \frac{x}{2} + \frac{x^2}{4}\right) \left(\frac{4}{x^2}\right) - 4 \left(1 + \frac{x}{2}\right) \left(\frac{8}{x^3}\right) + \frac{16}{x^4} \\ &= \left(1 + \frac{x}{2}\right)^4 - \frac{8}{x} \left(1 + \frac{x}{2}\right)^3 + \frac{24}{x^2} + \frac{24}{x} + 6 - \frac{32}{x^3} - \frac{16}{x^2} + \frac{16}{x^4} \\ &= \left(1 + \frac{x}{2}\right)^4 - \frac{8}{x} \left(1 + \frac{x}{2}\right)^3 + \frac{8}{x^2} + \frac{24}{x} + 6 - \frac{32}{x^3} + \frac{16}{x^4} \dots (1) \end{aligned}$$

Again by using Binomial Theorem, we obtain

$$\left(1 + \frac{x}{2}\right)^4 = {}^4C_0 (1)^4 + {}^4C_1 (1)^3 \left(\frac{x}{2}\right) + {}^4C_2 (1)^2 \left(\frac{x}{2}\right)^2 + {}^4C_3 (1)^3 \left(\frac{x}{2}\right)^3 + {}^4C_4 \left(\frac{x}{2}\right)^4$$

$$= 1 + 4 \times \frac{x}{2} + 6 \times \frac{x^2}{4} + 4 \times \frac{x^3}{8} + \frac{x^4}{16}$$

$$= 1 + 2x + \frac{3x^2}{2} + \frac{x^3}{2} + \frac{x^4}{16} \dots\dots (2)$$

$$\left(1 + \frac{x}{2}\right)^3 = {}^3C_0(1)^3 + {}^3C_1(1)^2\left(\frac{x}{2}\right) + {}^3C_2(1)\left(\frac{x}{2}\right)^2 + {}^3C_3\left(\frac{x}{2}\right)^3$$

$$= 1 + \frac{3x}{2} + \frac{3x^2}{4} + \frac{x^3}{8} \dots\dots (3)$$

From (1), (2) and (3), we obtain

$$\left[\left(1 + \frac{x}{2}\right) - \frac{2}{x}\right]^4$$

$$= 1 + 2x + \frac{3x^2}{2} + \frac{x^3}{2} + \frac{x^4}{16} - \frac{8}{x}\left(1 + \frac{3x}{2} + \frac{3x^2}{4} + \frac{x^3}{8}\right) + \frac{8}{x^2} + \frac{24}{x} + 6 - \frac{32}{x^3} + \frac{16}{x^4}$$

$$= 1 + 2x + \frac{3}{2}x^2 + \frac{x^3}{2} + \frac{x^4}{16} - \frac{8}{x} - 12 - 6x - x^2 + \frac{8}{x^2} + \frac{24}{x} + 6 - \frac{32}{x^3} + \frac{16}{x^4}$$

$$= \frac{16}{x} + \frac{8}{x^2} - \frac{32}{x^3} + \frac{16}{x^4} - 4x + \frac{x^2}{2} + \frac{x^3}{2} + \frac{x^4}{16} - 5$$

Question 10:

Find the expansion of $(3x^2 - 2ax + 3a^2)^3$ using binomial theorem.

Solution 10:

Using Binomial Theorem, the given expression $(3x^2 - 2ax + 3a^2)^3$ can be expanded as

$$\left[(3x^2 - 2ax) + 3a^2\right]^3$$

$$= {}^3C_0(3x^2 - 2ax)^3 + {}^3C_1(3x^2 - 2ax)^2(3a^2) + {}^3C_2(3x^2 - 2ax)(3a^2)^2 + {}^3C_3(3a^2)^3$$

$$= (3x^2 - 2ax)^3 + 3(9x^4 - 12ax^3 + 4a^2x^2)(3a^2) + 3(3x^2 - 2ax)(9a^4) + 27a^6$$

$$= (3x^2 - 2ax)^3 + 81a^2x^4 - 108a^3x^3 + 36a^4x^2 + 81a^4x^2 - 54a^5x + 27a^6$$

$$= (3x^2 - 2ax)^3 + 81a^2x^4 - 108a^3x^3 + 117a^4x^2 - 54a^5x + 27a^6 \dots\dots\dots (1)$$

Again by using Binomial Theorem, we obtain

$$(3x^2 - 2ax)^3$$

$$= {}^3C_0(3x^2)^3 - {}^3C_1(3x^2)^2(2ax) + {}^3C_2(3x^2)(2ax)^2 - {}^3C_3(2ax)^3$$

$$= 27x^6 - 3(9x^4)(2ax) + 3(3x^2)(4a^2x^2) - 8a^3x^3$$

$$= 27x^6 - 54ax^5 + 36a^2x^4 - 8a^3x^3 \dots\dots\dots (2)$$

From (1) and (2), we obtain

Chapter 8
Binomial Theorem

$$\begin{aligned} & (3x^2 - 2ax + 3a^2)^3 \\ &= 27x^6 - 54ax^5 + 36a^2x^4 - 8a^3x^3 + 81a^2x^4 - 108a^3x^3 + 117a^4x^2 - 54a^5x + 27a^6 \\ &= 27x^6 - 54ax^5 + 117a^2x^4 - 116a^3x^3 + 117a^4x^2 - 54a^5x + 27a^6 \end{aligned}$$