Exercise 2.1

Question 1:

Find the principal value of $\sin^{-1}\left(-\frac{1}{2}\right)$

Solution 1:

Let $\sin^{-1}\left(-\frac{1}{2}\right)=y$, Then $\sin y=\left(-\frac{1}{2}\right)=-\sin\left(\frac{\pi}{6}\right)=\sin\left(-\frac{\pi}{6}\right)$.

We know that the range of the principal value branch of $\sin^{-1}$ is $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

$and \sin\left(-\frac{\pi}{6}\right)=\frac{1}{2},$

Therefore, the principal value of $\sin^{-1}\left(-\frac{1}{2}\right) is -\frac{\pi}{6}$.

Question 2:

Find the principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

Solution 2:

Let $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)=y$. Then $\cos y=\frac{\sqrt{3}}{2}=\cos\left(\frac{\pi}{6}\right)$

We know that the range of the principal value branch of $\cos^{-1}$ is $[0,\pi]$

$and \cos\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$.

Therefore, the principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) is \frac{\pi}{6}$.

Question 3:

Find the principal value of $\text{cosec}^{-1}(2)$

Solution 3:

Let $\text{cosec}^{-1}(2) = y$.

Then, $\text{cosec } y = 2 = \text{cosec}\left(\frac{\pi}{6}\right)$.

We know that the range of the principal value branch of cosec^{-1} is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$.

Therefore, the principal value of $\text{cosec}^{-1}(2)$ is $\frac{\pi}{6}$.

Question 4:

Find the principal value of $\tan^{-1}\left(-\sqrt{3}\right)$

Solution 4:

Let $\tan^{-1}\left(-\sqrt{3}\right) = y$

Then, $\tan y = -\sqrt{3} = -\tan\frac{\pi}{3} = \tan\left(-\frac{\pi}{3}\right)$.

We know that the range of the principal value branch of $\tan^{-1}$ is $\left(-\frac{\pi}{2}\frac{\pi}{2}\right)$

$and \tan\left(-\frac{\pi}{3}\right) is -\sqrt{3}.$

Therefore, known that the principal value of $\tan^{-1}\left(-\sqrt{3}\right)$ is $-\frac{\pi}{3}$.

Question 5:

Find the principal value of $\cos^{-1}\left(-\frac{1}{2}\right)$

Solution 5:

$Let \cos^{-1}\left(-\frac{1}{2}\right) = y.$

Then, $\cos y = -\frac{1}{2} = -\cos\left(\frac{\pi}{3}\right) = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\left(\frac{2\pi}{3}\right)$.

We know that the range of the principal value branch of $\cos^{-1}$ is $[0, \pi]$

$and \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}.$

Therefore, the principal value of $\cos^{-1}\left(-\frac{1}{2}\right)$ is $\left(\frac{2\pi}{3}\right)$.

Question 6:

Find the principal value of $\tan^{-1}(-1)$

Solution 6:

Let $\tan^{-1}(-1) = \text{y}$.

Then, $\tan y = -1 = -\tan\left(\frac{\pi}{4}\right) = \tan\left(-\frac{\pi}{4}\right)$.

We know that the range of the principal value branch of $\tan^{-1}$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$and \tan\left(-\frac{\pi}{4}\right) = -1$.

Therefore, the principal value of $\tan^{-1}(-1)$ is $-\frac{\pi}{4}$.

Question 7:

Find the principal value of $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$

Solution 7:

Let $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = y$. Then, $\sec y = \frac{2}{\sqrt{3}} = \sec\left(\frac{\pi}{6}\right)$.

We know that the range of the principal value branch of $\sec^{-1}$ is $[0,\pi] - \left\{\frac{\pi}{2}\right\}$

$and \sec\left(\frac{\pi}{6}\right) = \frac{2}{\sqrt{3}}$.

Therefore, the principal value of $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right) is \frac{\pi}{6}$.

Question 8:

Find the principal value of $\cot^{-1}\left(\sqrt{3}\right)$

Solution 8:

Let $\cot^{-1}\left(\sqrt{3}\right)=y$. Then $\cot y=\sqrt{3}=\cot\left(\frac{\pi}{6}\right)$.

We know that the range of the principal value branch of $\cot^{-1}$ is $(0, \pi)$

and $\cot\left(\frac{\pi}{6}\right)=\sqrt{3}$.

Therefore, the principal value of $\cot^{-1}\left(\sqrt{3}\right)$ is $\frac{\pi}{6}$.

Question 9:

Find the principal value of $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

Solution 9:

Let $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)=y$.

Then $\cos y=-\frac{1}{\sqrt{2}}=-\cos\left(\frac{\pi}{4}\right)=\cos\left(\pi-\frac{\pi}{4}\right)=\cos\left(\frac{3\pi}{4}\right)$.

We know that the range of the principal value branch of $\cos^{-1}$ is $[0, \pi]$

and $\cos\left(\frac{3\pi}{4}\right)=-\frac{1}{\sqrt{2}}$.

Therefore, the principal value of $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ is $\frac{3\pi}{4}$.

Question 10:

Find the principal value of $\operatorname{cosec}^{-1}\left(-\sqrt{2}\right)$

Solution 10:

Let $\operatorname{cosec}^{-1}\left(-\sqrt{2}\right)=y$. Then, $\operatorname{cosec} y=-\sqrt{2}\ =-\operatorname{cosec}\left(\frac{\pi}{4}\right)=\operatorname{cosec}\left(-\frac{\pi}{4}\right)$.

We know that the range of the principal value branch of $\operatorname{cosec}^{-1}$ is $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\}$ and $\operatorname{cosec}\left(-\frac{\pi}{4}\right)=-\sqrt{2}$.

Therefore, the principal value of $\operatorname{cosec}^{-1}\left(-\sqrt{2}\right)$ is $-\frac{\pi}{4}$.

Question 11:

Find the value of $\tan^{-1}(1)+\cos^{-1}\left(-\frac{1}{2}\right)+\sin^{-1}\left(-\frac{1}{2}\right)$

Solution 11:

Let $\tan^{-1}(1)=x$.

Then, $\tan x=1=\tan\left(\frac{\pi}{4}\right)$.

$\therefore \tan^{-1}(1)=\frac{\pi}{4}$

Let $\cos^{-1}\left(-\frac{1}{2}\right)=y$.

Then, $\cos y=-\frac{1}{2}=-\cos\left(\frac{\pi}{3}\right)=\cos\left(\pi-\frac{\pi}{3}\right)=\cos\left(\frac{2\pi}{3}\right)$.

$\therefore\ \cos^{-1}\left(-\frac{1}{2}\right)=\frac{2\pi}{3}$

Let $\sin^{-1}\left(-\frac{1}{2}\right)=z$.

Then, $\sin z=-\frac{1}{2}=-\sin\left(\frac{\pi}{6}\right)=\sin\left(-\frac{\pi}{6}\right)$.

$\therefore \sin^{-1}\left(-\frac{1}{2}\right)=-\frac{\pi}{6}$

$\therefore \tan^{-1}(1)+\cos^{-1}\left(-\frac{1}{2}\right)+\sin^{-1}\left(-\frac{1}{2}\right)$

$$=\frac{\pi}{4}+\frac{2\pi}{3}-\frac{\pi}{6}$$

$$=\frac{3\pi+8\pi-2\pi}{12}=\frac{9\pi}{12}=\frac{3\pi}{4}$$

Question 12:

Find the value of $\cos^{-1}\left(\frac{1}{2}\right)+2\sin^{-1}\left(\frac{1}{2}\right)$

Solution 12:

$Let\cos^{-1}\left(\frac{1}{2}\right)=x.$

Then, $\cos x=\frac{1}{2}=\cos\left(\frac{\pi}{3}\right).$

$\therefore\cos^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{3}$

$Let\sin^{-1}\left(\frac{1}{2}\right)=y.$

Then, $\sin y=\frac{1}{2}=\sin\left(\frac{\pi}{6}\right).$

$\therefore\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{6}$

$$\therefore\cos^{-1}\left(\frac{1}{2}\right)+2\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{3}+2\times\frac{\pi}{6}=\frac{\pi}{3}+\frac{\pi}{3}=\frac{2\pi}{3}$$

Question 13:

If $\sin^{-1}x=y$, then

(A) $0\le y\le\pi$ **(B)** $-\frac{\pi}{2}\le y\le\frac{\pi}{2}$

(C) $0<y<\pi$ **(D)** $-\frac{\pi}{2}<y<\frac{\pi}{2}$

Solution 13:

It is given that $\sin^{-1} x = y$.

We know that the range of the principal value branch of $\sin^{-1}$ $is \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Therefore, $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

Answer choice (B) is correct.

Question 14:

$\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$ is equal to

(A) π **(B)** $-\pi/3$
(C) $\pi/3$ **(D)** $2\pi/3$

Solution 14:

Let $\tan^{-1}\sqrt{3} = x.$.

Then, $\tan x = \sqrt{3} = \tan\frac{\pi}{3}$

We know that the range of the principal value branch of $\tan^{-1}$ is $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

$\therefore \tan^{-1}\sqrt{3} = \frac{\pi}{3}$

Let $\sec^{-1}(-2) = y.$

Then, $\sec y = -2 = -\sec\left(\frac{\pi}{3}\right) = \sec\left(\pi - \frac{\pi}{3}\right) = \sec\frac{2\pi}{3}$.

We know that the range of the principal value branch of $\sec^{-1}$ is $[0, \pi] - \left\{\frac{\pi}{2}\right\}$.

$\therefore \sec^{-1}(-2) = \frac{2\pi}{3}$

Thus,

$\tan^{-1}(\sqrt{3}) - \sec^{-1}(-2)$

$= \frac{\pi}{3} - \frac{2\pi}{3} = -\frac{\pi}{3}$

Exercise 2.2

Question 1:

Prove $3\sin^{-1}x = \sin^{-1}(3x - 4x^3), x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

Solution 1:

To Prove $3\sin^{-1}x = \sin^{-1}(3x - 4x^3), \textit{where } x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

Let $x = \sin\theta.\ Then, \sin^{-1}x = \theta.$

We have,

R.H.S

$\sin^{-1}(3x - 4x^3) = \sin^{-1}(3\sin\theta - 4\sin^3\theta)$

$= \sin^{-1}(\sin 3\theta)$

$= 3\theta$

$= 3\sin^{-1}x = L.H.S$

Question 2:

Prove $3\cos^{-1}x = \cos^{-1}(4x^3 - 3x), x \in \left[\frac{1}{2}, 1\right]$

Solution 2:

To Prove $3\cos^{-1}x = \cos^{-1}(4x^3 - 3x), x \in \left[\frac{1}{2}, 1\right]$

Let $x = \cos\theta$. Then, $\cos^{-1}x = \theta$

We have

$R.H.S$

$\cos^{-1}(4x^3 - 3x)$

$= \cos^{-1}\left(4\cos^3\theta - 3\cos\theta\right)$

$= \cos^{-1}\left(\cos 3\theta\right)$

$= 3\theta$

$= 3\cos^{-1} x = L.H.S$

Question 3:

Prove $\tan^{-1}\left(\frac{2}{11}\right) + \tan^{-1}\left(\frac{7}{24}\right) = \tan^{-1}\left(\frac{1}{2}\right)$

Solution 3:

To prove: $\tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24} = \tan^{-1}\frac{1}{2}$

$L.H.S.$

$\tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24}$

$= \tan^{-1}\frac{\frac{2}{11}+\frac{7}{24}}{1-\left(\frac{2}{11}.\frac{7}{24}\right)}$ $\left[\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\right]$

$= \tan^{-1}\frac{11\times 24}{\frac{11\times 24\ -14}{11\times 24}}$

$= \tan^{-1}\frac{48+77}{264-14}$

$= \tan^{-1}\left(\frac{125}{250}\right) = \tan^{-1}\left(\frac{1}{2}\right) = \text{R.H.S.}$

Question 4:

Prove $2\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{31}{17}$

Solution 4:

To prove: $2\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{31}{17}$

$$\text{L.H.S} = 2\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{7}$$

$$=\tan^{-1}\frac{2.\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2}+\tan^{-1}\frac{1}{7}\left[2\tan^{-1}x=\tan^{-1}\frac{2x}{1-x^2}\right]$$

$$=\tan^{-1}\frac{1}{\left(\frac{3}{4}\right)}+\tan^{-1}\frac{1}{7}$$

$$=\tan^{-1}\frac{4}{3}+\tan^{-1}\frac{1}{7}$$

$$=\tan^{-1}\frac{\frac{4}{3}+\frac{1}{7}}{1-\frac{4}{3}.\frac{1}{7}}\left[\tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}\right]$$

$$=\tan^{-1}\left(\frac{\frac{28+3}{21}}{\frac{21-4}{21}}\right)$$

$$=\tan^{-1}\frac{31}{17}=\text{R.H.S.}$$

Question 5:

Write the function in the simplest form: $\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}, x\neq 0$

Solution 5:

$$\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$$

Put $x=\tan\theta \Rightarrow \theta=\tan^{-1}x$

$\therefore \tan^{-1}\frac{\sqrt{1+x^2}-1}{x}=\tan^{-1}\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}$

$=\tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)=\tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$

$\tan^{-1}\left(\frac{2\sin^2\left(\frac{\theta}{2}\right)}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$

$=\tan^{-1}\left(\tan\frac{\theta}{2}\right)=\frac{\theta}{2}=\frac{1}{2}\tan^{-1}x$

Question 6:

Write the function in the simplest form: $\tan^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right),|x|>1$

Solution 6:

$\tan^{-1}\frac{1}{\sqrt{x^2-1}},|x|>1$

Put $x=\text{cosec}\,\theta \Rightarrow \theta=\cos ec^{-1}x$

$\therefore \tan^{-1}\frac{1}{\sqrt{x^2-1}}$

$=\tan^{-1}\frac{1}{\sqrt{\cos ec^2\theta-1}}$

$=\tan^{-1}\left(\frac{1}{\cot\theta}\right)$

$=\tan^{-1}(\tan\theta)$

$=\theta$

$=\cos ec^{-1}x$

$=\frac{\pi}{2}-\sec^{-1}x \quad \left[As, \cos ec^{-1}x+\sec^{-1}x=\frac{\pi}{2}\right]$

Question 7:

Write the function in the simplest form: $\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right), x < \pi$

Solution 7:

$$\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right),\ x < \pi$$

$$\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)$$

$$= \tan^{-1}\left(\sqrt{\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}}\right)$$

$$= \tan^{-1}\left(\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right)$$

$$= \tan^{-1}\left(\tan\frac{x}{2}\right)$$

$$= \frac{x}{2}$$

Question 8:

Write the function in the simplest form: $\tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right), 0 < x < \pi$

Solution 8:

$$\tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)$$

$$= \tan^{-1}\left(\frac{1-\left(\frac{\sin x}{\cos x}\right)}{1+\left(\frac{\sin x}{\cos x}\right)}\right)$$

$$= \tan^{-1}\left(\frac{1-\tan x}{1+\tan x}\right)$$

$$= \tan^{-1}(1) - \tan^{-1}(\tan x) \qquad \left[\because \quad \frac{-y}{\cdot xy} = \tan^{-1} x - \tan^{-1} y\right]$$

$$= \frac{\pi}{4} - x$$

Question 9:

Write the function in the simplest form: $\tan^{-1}\frac{x}{\sqrt{a^2-x^2}}, |x| < a$

Solution 9:

$$\tan^{-1}\frac{x}{\sqrt{a^2-x^2}}$$

Let, $x = a\sin\theta \Rightarrow \frac{x}{a} = \sin\theta \Rightarrow \sin^{-1}\left(\frac{x}{a}\right)$

$$\therefore \tan^{-1}\frac{x}{\sqrt{a^2-x^2}}$$

$$= \tan^{-1}\left(\frac{a\sin\theta}{\sqrt{a^2-a^2\sin^2\theta}}\right)$$

$$= \tan^{-1}\left(\frac{a\sin\theta}{a\sqrt{1-\sin^2\theta}}\right)$$

$$= \tan^{-1}\left(\frac{a\sin\theta}{a\cos\theta}\right)$$

$$= \tan^{-1}(\tan\theta) = \theta = \sin^{-1}\frac{x}{a}$$

Question 10:

Write the function in the simplest form: $\tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right), a > 0; \frac{-a}{\sqrt{3}} \le x \le \frac{a}{\sqrt{3}}$

Solution 10:

Consider, $\tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right)$

Let

$$x=a\tan\theta \Rightarrow \frac{x}{a}=\tan\theta \Rightarrow \theta=\tan^{-1}\left(\frac{x}{a}\right)$$

$$\tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right)$$

$$=\tan^{-1}\left(\frac{3a^2\cdot a\tan\theta-a^3\tan^3\theta}{a^3-3a\cdot a^2\tan^2\theta}\right)$$

$$=\tan^{-1}\left(\frac{3a^3\tan\theta-a^3\tan^3\theta}{a^3-3a^3\tan^2\theta}\right)$$

$$=\tan^{-1}(\tan 3\theta)$$

$$=3\theta$$

$$=3\tan^{-1}\frac{x}{a}$$

Question 11:

Find the value of $\tan^{-1}\left[2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right]$

Solution 11:

Let $\sin^{-1}\frac{1}{2}=x.$

$$Then, \sin x=\frac{1}{2}=\sin\left(\frac{\pi}{6}\right).$$

$$\therefore \sin^{-1}\frac{1}{2}=\frac{\pi}{6}$$

$$\therefore \tan^{-1}\left[2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right]$$

$$=\tan^{-1}\left[2\cos\left(2\times\frac{\pi}{6}\right)\right]$$

$$= \tan^{-1}\left[2\cos\frac{\pi}{3}\right]$$

$$= \tan^{-1}\left[2\times\frac{1}{2}\right]$$

$$= \tan^{-1}1 = \frac{\pi}{4}$$

Question 12:

Find the value of $\cot\left(\tan^{-1}a + \cot^{-1}a\right)$

Solution 12:

$$\cot\left(\tan^{-1}a + \cot^{-1}a\right)$$

$$= \cot\left(\frac{\pi}{2}\right) \qquad \left[\because \quad \vdash\cot^{-1}x = \frac{\pi}{2}\right]$$

$$= 0$$

Question 13:

Find the value of $\tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2} + \cos^{-1}\frac{1-y^2}{1+y^2}\right], |x| < 1, y > 0$ and $xy < 1$

Solution 13:

Let $x = \tan\theta$.

$Then, \theta = \tan^{-1}x.$

$$\therefore \sin^{-1}\frac{2x}{1+x^2} = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$$= \sin^{-1}\left(\sin 2\theta\right)$$

$$= 2\theta$$

$$= 2\tan^{-1}x$$

Let $y = \tan\theta. Then, \theta = \tan^{-1}y.$

$$\therefore \cos^{-1}\left(\frac{1-y^2}{1+y^2}\right)$$

$$=\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

$$=\cos^{-1}\left(\cos 2\theta\right)$$

$$=2\theta=2\tan^{-1}y$$

$$\therefore \tan\frac{1}{2}\left[\sin^{-1}\left(\frac{2x}{1+x^2}\right)+\cos^{-1}\left(\frac{1-y^2}{1+y^2}\right)\right]$$

$$=\tan\frac{1}{2}\left[2\tan^{-1}x+2\tan^{-1}y\right] \qquad \left[As, \tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}\right]$$

$$=\tan\left[\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$$

$$=\frac{x+y}{1-xy}$$

Question 14:

If $\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right)=1$, then find the value of x.

Solution 14:

$$\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right)=1$$

$$\Rightarrow \sin\left(\sin^{-1}\frac{1}{5}\right)\cos\left(\cos^{-1}x\right)+\cos\left(\sin^{-1}\frac{1}{5}\right)\sin\left(\cos^{-1}x\right)=1$$

$$\left[\because \qquad B)=\sin A\cdot\cos B+\cos A\cdot\sin B\right]$$

$$\Rightarrow \frac{1}{5}\cdot x+\cos\left(\sin^{-1}\frac{1}{5}\right)\sin\left(\cos^{-1}x\right)=1$$

$$\Rightarrow \frac{x}{5}+\cos\left(\sin^{-1}\frac{1}{5}\right)\sin\left(\cos^{-1}x\right)=1 \qquad ...(1)$$

Now, let $\sin^{-1}\frac{1}{5}=y$

Then,

$$\sin^{-1}\frac{1}{5}=y$$

$$\sin y=\frac{1}{5}$$

$$\Rightarrow \cos y=\sqrt{1-\left(\frac{1}{5}\right)^2}=\frac{2\sqrt{6}}{5}$$

$$\Rightarrow y=\cos^{-1}\left(\frac{2\sqrt{6}}{5}\right)$$

$$\therefore \sin^{-1}\frac{1}{5}=\cos^{-1}\left(\frac{2\sqrt{6}}{5}\right) \quad ...(2)$$

Let $\cos^{-1}x=z$.

Then, $\cos z=x$

$$\Rightarrow \sin z=\sqrt{1-x^2}$$

$$\Rightarrow z=\sin^{-1}\left(\sqrt{1-x^2}\right).$$

$$\therefore \cos^{-1}x=\sin^{-1}\left(\sqrt{1-x^2}\right) \quad ...(3)$$

From (1), (2) and (3) we have:

$$\frac{x}{5}+\cos\left(\cos^{-1}\frac{2\sqrt{6}}{5}\right)\cdot\sin\left(\sin^{-1}\sqrt{1-x^2}\right)=1$$

$$\Rightarrow \frac{x}{5}+\frac{2\sqrt{6}}{5}\cdot\sqrt{1-x^2}=1$$

$$\Rightarrow x+2\sqrt{6}\sqrt{1-x^2}=5$$

$$\Rightarrow 2\sqrt{6}\sqrt{1-x^2}=5-x$$

On squaring both sides, we get:

$$\left(2\sqrt{6}\right)^2\left(1-x^2\right)=25+x^2-10x$$

$$\Rightarrow \left(4\times 6\right)\left(1-x^2\right)=25+x^2-10x$$

$$\Rightarrow 24-24x^2=25+x^2-10x$$

$$\Rightarrow 25x^2-10x+1=0$$

$$\Rightarrow \left(5x-1\right)^2=0$$

$$\Rightarrow \left(5x-1\right)=0$$

$$\Rightarrow x=\frac{1}{5}$$

Hence, the value of x is $\frac{1}{5}$.

Question 15:

If $\tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$, then find the value of x.

Solution 15:

$$\tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\left(\frac{x-1}{x+2}\right)\left(\frac{x+1}{x+2}\right)}\right]=\frac{\pi}{4} \qquad \left[As, \tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}\right]$$

$$\Rightarrow \tan^{-1}\left[\frac{\left(x-1\right)\left(x+2\right)+\left(x+1\right)\left(x-2\right)}{\left(x+2\right)\left(x-2\right)-\left(x-1\right)\left(x+1\right)}\right]=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left[\frac{x^2+x-2+x^2-x-2}{x^2-4-x^2+1}\right]=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left[\frac{2x^2-4}{-3}\right]=\frac{\pi}{4}$$

$$\Rightarrow \tan\left[\tan^{-1}\frac{4-2x^2}{3}\right]=\tan\frac{\pi}{4}$$

$$\Rightarrow \frac{4-2x^2}{3}=1$$

$\Rightarrow 4-2x^2=3$

$\Rightarrow 2x^2=4-3=1$

$\Rightarrow x=\pm\frac{1}{\sqrt{2}}$

Hence, the value of x is $\pm\frac{1}{\sqrt{2}}$.

Question 16:

Find the values of $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$

Solution 16:

Consider, $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$

We know that $\sin^{-1}(\sin x)=x$

If $x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, which is the principal value branch of $\sin^{-1}x$..

Here, $\frac{2\pi}{3}\notin\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$

Now, $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$ can be written as:

$\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$

$=\sin^{-1}\left[\sin\left(\pi-\frac{2\pi}{3}\right)\right]$

$=\sin^{-1}\left(\sin\frac{\pi}{3}\right),\quad where\ \frac{\pi}{3}\in\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$

$\therefore \sin^{-1}\left(\sin\frac{2\pi}{3}\right)=\sin^{-1}\left[\sin\frac{\pi}{3}\right]=\frac{\pi}{3}$

Question 17:

Find the values of $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$

Solution 17:

Consider, $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$

We know that $\tan^{-1}(\tan x)=x$

If $x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, which is the principal value branch of $\tan^{-1}x$.

Here, $\frac{3\pi}{4}\notin\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

Now, $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$ can be written as:

$$\tan^{-1}\left(\tan\frac{3\pi}{4}\right)=\tan^{-1}\left[-\tan\left(\frac{-3\pi}{4}\right)\right]=\tan^{-1}\left[-\tan\left(\pi-\frac{\pi}{4}\right)\right]$$

$$\tan^{-1}\left(-\tan\frac{\pi}{4}\right)=\tan^{-1}\left[\tan\left(\frac{-\pi}{4}\right)\right]\text{ where }-\frac{\pi}{4}\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$$

$$\therefore\tan^{-1}\left(\tan\frac{3\pi}{4}\right)=\tan^{-1}\left[\tan\left(\frac{-\pi}{4}\right)\right]=\frac{-\pi}{4}$$

Question 18:

Find the values of $\tan\left(\sin^{-1}\frac{3}{5}+\cot^{-1}\frac{3}{2}\right)$

Solution 18:

Let $\sin^{-1}\frac{3}{5}=x$.

Then ,

$$\sin x=\frac{3}{5}$$

$$\Rightarrow\cos x=\sqrt{1-\sin^2 x}=\frac{4}{5}$$

$$\Rightarrow\sec x=\frac{5}{4}$$

$$\therefore \tan x = \sqrt{\sec^2 x - 1} = \sqrt{\frac{25}{16} - 1} = \frac{3}{4}$$

$$\therefore x = \tan^{-1}\frac{3}{4}$$

$$\therefore \sin^{-1}\frac{3}{5} = \tan^{-1}\frac{3}{4} \qquad ...(i)$$

Now, $\cot^{-1}\frac{3}{2} = \tan^{-1}\frac{2}{3} \qquad ...(ii)$ $\left[\because \quad = \cot^{-1} x\right]$

Therefore, $\tan\left(\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right)$

$$= \tan\left(\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{2}{3}\right)$$ [Using (i) and (ii)]

$$= \tan\left[\tan^{-1}\left(\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\cdot\frac{2}{3}}\right)\right]$$ $\left[As, \tan^{-1} x + \tan^{-1} y = \tan^{-1}\frac{x+y}{1-xy}\right]$

$$= \tan\left(\tan^{-1}\frac{9+8}{12-6}\right)$$

$$= \tan\left(\tan^{-1}\frac{17}{6}\right) = \frac{17}{6}$$

Question 19:

Find the values of $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$ is equal to

$(A)\ \frac{7\pi}{6}$ $\qquad (B)\ \frac{5\pi}{6}$

$(C)\ \frac{\pi}{3}$ $\qquad (D)\ \frac{\pi}{6}$

Solution 19:

We know that $\cos^{-1}(\cos x) = x$ if $x \in [0, \pi]$, which is the principal value branch of $\cos^{-1} x$.

Here, $\frac{7\pi}{6} \notin [0, \pi]$.

Now, $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$ can be written as :

$$\cos^{-1}\left(\cos\frac{7\pi}{6}\right)=\cos^{-1}\left(\cos\frac{-7\pi}{6}\right)=\cos^{-1}\left[\cos\left(2\pi-\frac{7\pi}{6}\right)\right] \qquad \left[\because \qquad +x)=\cos x\right]$$

$$\therefore \cos^{-1}\left(\cos\frac{7\pi}{6}\right)=\cos^{-1}\left(\cos\frac{5\pi}{6}\right)=\frac{5\pi}{6}$$

The correct answer is B.

Question 20:

Find the values of $\sin\left(\frac{\pi}{3}-\sin^{-1}\left(-\frac{1}{2}\right)\right)$ is equal to

(A) $\frac{1}{2}$ **(B)** $\frac{1}{3}$

(C) $\frac{1}{4}$ **(D)** 1

Solution 20:

Let $\sin^{-1}\left(\frac{-1}{2}\right)=x$.

Then , $\sin x=\frac{-1}{2}=-\sin\frac{\pi}{6}=\sin\left(\frac{-\pi}{6}\right)$.

We know that the range of the principal value branch of $\sin^{-1} is\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$.

$$\sin^{-1}\left(\frac{-1}{2}\right)=\frac{\pi}{6}$$

$$\therefore \sin\left(\frac{\pi}{3}-\sin^{-1}\left(\frac{-1}{2}\right)\right)=\sin\left(\frac{\pi}{3}+\frac{\pi}{6}\right)=\sin\left(\frac{3\pi}{6}\right)=\sin\left(\frac{\pi}{2}\right)=1$$

The correct answer is D.

Miscellaneous Exercise

Question 1:

Find the value of $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$

Solution 1:

We Know that $\cos^{-1}(\cos x) = x$ if $x \in [0,\pi]$, which is the principal value branch of $\cos^{-1} x$.

Here, $\frac{13\pi}{6} \notin [0,\pi]$.

Now, $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$ can be written as :

$$\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$$

$$= \cos^{-1}\left[\cos\left(2\pi + \frac{\pi}{6}\right)\right]$$

$$= \cos^{-1}\left[\cos\left(\frac{\pi}{6}\right)\right] \qquad where \frac{\pi}{6} \in [0,\pi].$$

$$\therefore \cos^{-1}\left(\cos\frac{13\pi}{6}\right) = \cos^{-1}\left[\cos\left(\frac{\pi}{6}\right)\right] = \frac{\pi}{6}$$

Question 2:

Find the value of $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$

Solution 2:

We know that $\tan^{-1}(\tan x) = x$ if $x \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, which is the principal value branch of $\tan^{-1} x$.

Here, $\frac{7\pi}{6} \notin \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

Now, $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$ can be written as:

$$\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$$

$$= \tan^{-1}\left[\tan\left(2\pi - \frac{5\pi}{6}\right)\right] \qquad \left[\because \qquad -x\right) = -\tan x\right]$$

$$= \tan^{-1}\left[-\tan\left(\frac{5\pi}{6}\right)\right] \qquad \left[\because \qquad -x\right) = -\tan x\right]$$

$$= \tan^{-1}\left[\tan\left(\pi - \frac{5\pi}{6}\right)\right]$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{6}\right)\right], \text{ where } \frac{\pi}{6} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\therefore \tan^{-1}\left(\tan\frac{7\pi}{6}\right) = \tan^{-1}\left(\tan\frac{\pi}{6}\right) = \frac{\pi}{6}$$

Question 3:

Prove $2\sin^{-1}\frac{3}{5} = \tan^{-1}\frac{24}{7}$

Solution 3:

Let $\sin^{-1}\frac{3}{5} = x.$ Then $\sin x = \frac{3}{5}.$

$$\Rightarrow \cos x = \sqrt{1-\left(\frac{3}{5}\right)^2} = \frac{4}{5}$$

$$\therefore \tan x = \frac{3}{4}$$

$$\therefore x = \tan^{-1}\frac{3}{4} \Rightarrow \sin^{-1}\frac{3}{5} = \tan^{-1}\frac{3}{4}$$

Now, we have:

L.H.S

$$2\sin^{-1}\frac{3}{5} = 2\tan^{-1}\frac{3}{4}$$

$$= \tan^{-1}\left(\frac{2\times\frac{3}{4}}{1-\left(\frac{3}{4}\right)^2}\right) \qquad \left[\because \qquad = \tan^{-1}\frac{2x}{1-x^2}\right]$$

$$= \tan^{-1}\left(\frac{\frac{3}{2}}{\frac{16-9}{16}}\right) = \tan^{-1}\left(\frac{3}{2}\times\frac{16}{7}\right)$$

$$= \tan^{-1}\frac{24}{7} = R.H.S.$$

Question 4:

Prove $\sin^{-1}\frac{8}{17}+\sin^{-1}\frac{3}{5}=\tan^{-1}\frac{77}{36}$

Solution 4:

$\sin^{-1}\frac{8}{17}=x.$

Then, $\sin x=\frac{8}{17}\Rightarrow \cos x=\sqrt{1-\left(\frac{8}{17}\right)^2}=\sqrt{\frac{225}{289}}=\frac{15}{17}.$

$\therefore \tan x=\frac{8}{15}\Rightarrow x=\tan^{-1}\frac{8}{15}$

$\therefore \sin^{-1}\frac{8}{17}=\tan^{-1}\frac{8}{15}$ .....(1)

Now, let $\sin^{-1}\frac{3}{5}=y$

Then, $\sin y=\frac{3}{5}\Rightarrow \cos y=\sqrt{1-\left(\frac{3}{5}\right)^2}=\sqrt{\frac{16}{25}}=\frac{4}{5}.$

$\therefore \tan y=\frac{3}{5}\Rightarrow y=\tan^{-1}\frac{3}{4}$

$\therefore \sin^{-1}\frac{3}{5}=\tan^{-1}\frac{3}{4}$ ......(2)

Now, we have:

L.H.S.

$\sin^{-1}\frac{8}{17}+\sin^{-1}\frac{3}{5}$

Using (1) and (2), we get

$$= \tan^{-1}\frac{8}{15} + \tan^{-1}\frac{3}{4}$$

$$= \tan^{-1}\frac{\frac{8}{15}+\frac{3}{4}}{1-\frac{8}{15}\times\frac{3}{4}} \qquad \left[\because \qquad + \tan^{-1} y = \tan^{-1}\frac{x+y}{1-xy}\right]$$

$$= \tan^{-1}\left(\frac{32+45}{60-24}\right)$$

$$= \tan^{-1}\frac{77}{36} = R.H.S.$$

Question 5:

Prove $\cos^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} = \cos^{-1}\frac{33}{65}$

Solution 5:

Let $\cos^{-1}\frac{4}{5} = x$

Then, $\cos x = \frac{4}{5} \Rightarrow \sin x = \sqrt{1-\left(\frac{4}{5}\right)^2} = \frac{3}{5}$

$\therefore \tan x = \frac{3}{4} \Rightarrow x = \tan^{-1}\frac{3}{4}$

$\therefore \cos^{-1}\frac{4}{5} = \tan^{-1}\frac{3}{4}$ ....(1)

Now, let $\cos^{-1}\frac{12}{13} = y.$

Then, $\cos y = \frac{12}{13} \Rightarrow \sin y = \frac{5}{13}.$

$\therefore \tan y = \frac{5}{12} \Rightarrow y = \tan^{-1}\frac{5}{12}$

$\therefore \cos^{-1}\frac{12}{13} = \tan^{-1}\frac{5}{12}$ ....(2)

Let $\cos^{-1}\frac{33}{65} = z$

Then, $\cos z = \frac{33}{65} \Rightarrow \sin z = \frac{56}{65}$

$\therefore \tan z = \frac{56}{33} \Rightarrow z = \tan^{-1}\frac{56}{33}$

$\therefore \cos^{-1}\frac{33}{65} = \tan^{-1}\frac{56}{33}$ ...(3)

Now,

$L.H.S.$

$\cos^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13}$

Using (1) and (2), we get

$$= \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{5}{12}$$

$$= \tan^{-1}\left(\frac{\frac{3}{4}+\frac{5}{12}}{1-\left(\frac{3}{4}\times\frac{5}{12}\right)}\right) \qquad \left[\because \quad \vdash \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$$

$$= \tan^{-1}\left(\frac{36+20}{48-15}\right)$$

$$= \tan^{-1}\frac{56}{33}$$

$$= \cos^{-1}\frac{56}{33}$$

$= R.H.S.$

Question 6:

Prove $\cos^{-1}\frac{12}{13} + \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{56}{65}$

Solution 6:

Let $\sin^{-1}\frac{3}{5} = x$. Then, $\sin x = \frac{3}{5} \Rightarrow \cos x = \sqrt{1-\left(\frac{3}{5}\right)^2} = \sqrt{\frac{16}{25}} = \frac{4}{5}$.

$\therefore \tan x = \frac{3}{4} \Rightarrow x = \tan^{-1}\frac{3}{4}$

$\therefore \sin^{-1}\frac{3}{5} = \tan^{-1}\frac{3}{4} \qquad ...(1)$

Now, let $\cos^{-1}\frac{12}{13} = y$

$Then, \cos y = \frac{12}{13} \Rightarrow \sin y = \frac{5}{13}.$

$\therefore \tan y = \frac{5}{12} \Rightarrow y = \tan^{-1}\frac{5}{12}$

$\therefore \cos^{-1}\frac{12}{13} = \tan^{-1}\frac{5}{12} \qquad ...(2)$

Let $\sin^{-1}\frac{56}{65} = z.$

Then, $\sin z = \frac{56}{65} \Rightarrow \cos z = \frac{33}{65}.$

$\therefore \tan z = \frac{56}{33} \Rightarrow z = \tan^{-1}\frac{56}{33}$

$\therefore \sin^{-1}\frac{56}{65} = \tan^{-1}\frac{56}{33} \qquad ...(3)$

Now, we have

L.H.S

$\cos^{-1}\frac{12}{13} + \sin^{-1}\frac{3}{5}$

$= \tan^{-1}\frac{5}{12} + \tan^{-1}\frac{3}{4}$

Using (1) and (2), we get

$$=\tan^{-1}\frac{\frac{5}{12}+\frac{3}{4}}{1-\left(\frac{5}{12}\cdot\frac{3}{4}\right)} \qquad \left[\because \quad \tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}\right]$$

$$=\tan^{-1}\left(\frac{20+36}{48-15}\right)$$

$$=\tan^{-1}\frac{56}{33}$$

$$=\sin^{-1}\frac{56}{65} \qquad \text{[Using (3)]}$$

$$=R.H.S$$

Question 7:

Prove $\tan^{-1}\frac{63}{16}=\sin^{-1}\frac{5}{13}+\cos^{-1}\frac{3}{5}$

Solution 7:

Let $\sin^{-1}\frac{5}{13}=x$

Then, $\sin x=\frac{5}{13}\Rightarrow\cos x=\frac{12}{13}$.

$\therefore \tan x=\frac{5}{12}\Rightarrow x=\tan^{-1}\frac{5}{12}$

$\therefore \sin^{-1}\frac{5}{13}=\tan^{-1}\frac{5}{12}$ …..(1)

Let $\cos^{-1}\frac{3}{5}=y$.

Then, $\cos y=\frac{3}{5}\Rightarrow\sin y=\frac{4}{5}$.

$Thus, \tan y=\frac{4}{3}\Rightarrow y=\tan^{-1}\frac{4}{3}$

$\therefore \cos^{-1}\frac{3}{5}=\tan^{-1}\frac{4}{3}$ ….(2)

Using (1) and (2), we have

R.H.S.

$$\sin^{-1}\frac{5}{12}+\cos^{-1}\frac{3}{5}$$

$$= \tan^{-1}\frac{5}{12}+\tan^{-1}\frac{4}{3}$$

$$= \tan^{-1}\left(\frac{\frac{5}{12}+\frac{4}{3}}{1-\frac{5}{12}\times\frac{4}{3}}\right) \qquad \left[\because \qquad + \tan^{-1} y = \tan^{-1}\frac{x+y}{1-xy}\right]$$

$$= \tan^{-1}\left(\frac{15+48}{36-20}\right)$$

$$= \tan^{-1}\frac{63}{16}$$

$$= \text{L.H.S}.$$

Question 8:

Prove $\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$

Solution 8:

L.H.S

$$\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{8}$$

$$= \tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{7}}{1-\frac{1}{5}\times\frac{1}{7}}\right)+\tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{8}}{1-\frac{1}{3}\times\frac{1}{8}}\right) \qquad \left[\tan^{-1} x+\tan^{-1} y=\tan^{-1}\frac{x+y}{1-xy}\right]$$

$$= \tan^{-1}\left(\frac{7+5}{35-1}\right)+\tan^{-1}\left(\frac{8+3}{24-1}\right)$$

$$= \tan^{-1}\frac{12}{34}+\tan^{-1}\frac{11}{23}$$

$$= \tan^{-1}\frac{6}{17}+\tan^{-1}\frac{11}{23}$$

$$= \tan^{-1}\left(\frac{\frac{6}{17}+\frac{11}{23}}{1-\frac{6}{17}\times\frac{11}{23}}\right)$$

$= \tan^{-1}\left(\frac{325}{325}\right) = \tan^{-1} 1$

$= \frac{\pi}{4} = \text{R.H.S.}$

Question 9:

Prove $\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right), x \in [0.1]$

Solution 9:

Let $x = \tan^2\theta$.

Then $\sqrt{x} = \tan\theta \Rightarrow \theta = \tan^{-1}\sqrt{x}$.

$\therefore \frac{1-x}{1+x} = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \cos 2\theta$

Now, we have:

$R.H.S$

$\frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)$

$= \frac{1}{2}\cos^{-1}(\cos 2\theta)$

$= \frac{1}{2} \times 2\theta = \theta = \tan^{-1}\sqrt{x} = L.H.S.$

Question 10:

Prove $\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) = \frac{x}{2}, x \in \left(0, \frac{\pi}{4}\right)$

Solution 10:

Consider $\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}$

$$=\frac{\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)^2}{\left(\sqrt{1+\sin x}-\sqrt{1-\sin x}\right)^2} \qquad \text{(by rationalizing)}$$

$$=\frac{(1+\sin x)+(1-\sin x)+2\sqrt{(1+\sin x)(1-\sin x)}}{1+\sin x-1+\sin x}$$

$$=\frac{2\left(1+\sqrt{1-\sin^2 x}\right)}{2\sin x}$$

$$=\frac{1+\cos x}{\sin x}$$

$$=\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$=\cot\frac{x}{2}$$

$$\therefore L.H.S. = \cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$$

$$=\cot^{-1}\left(\cot\frac{x}{2}\right)=\frac{x}{2}=R.H.S.$$

Question 11:

Prove $\tan^{-1}\left(\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)=\dfrac{\pi}{4}-\dfrac{1}{2}\cos^{-1}x,\ -\dfrac{1}{\sqrt{2}}\le x\le 1$

[**Hint** : put $x=\cos 2\theta$]

Solution 11:

Let, $x=\cos 2\theta$ then, $\theta=\dfrac{1}{2}\cos^{-1}x.$

Thus, we have:

$$L.H.S. = \tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{2\cos^2 2\theta}-\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2 2\theta}+\sqrt{2\sin^2\theta}}\right)$$

$$= \tan^{-}\left(\frac{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta+\sqrt{2}\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right)$$

$$= \tan^{-1}1-\tan^{-1}(\tan\theta) \qquad \left[\because \quad \left(\frac{x-y}{1+xy}\right)=\tan^{-1}x-\tan^{-1}y\right]$$

$$= \frac{\pi}{4}-\theta=\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x = \text{R.H.S.}$$

Question 12:

Prove $\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\frac{1}{3}=\frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}$

Solution 12:

$$\text{L.H.S.} = \frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\frac{1}{3}$$

$$= \frac{9}{4}\left(\frac{\pi}{2}-\sin^{-1}\frac{1}{3}\right)$$

$$= \frac{9}{4}\left(\cos^{-1}\frac{1}{3}\right) \qquad \ldots\ldots(1) \qquad \left[\because \quad \cos^{-1}x=\frac{\pi}{2}\right]$$

Now, let $\cos^{-1}\frac{1}{3}=x$

Then, $\cos x = \frac{1}{3} \Rightarrow \sin x = \sqrt{1-\left(\frac{1}{3}\right)^2} = \frac{2\sqrt{2}}{3}$.

$\therefore x = \sin^{-1}\frac{2\sqrt{2}}{3} \Rightarrow \cos^{-1}\frac{1}{3} \Rightarrow \sin^{-1}\frac{2\sqrt{2}}{3}$

$\therefore$ L.H.S. $= \frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}$ = R.H.S.

Question 13:

Solve $2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$

Solution 13:

$2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$

$\Rightarrow \tan^{-1}\left(\frac{2\cos x}{1-\cos^2 x}\right) = \tan^{-1}(2\operatorname{cosec} x) \qquad \left[\because \quad c = \tan^{-1}\frac{2x}{1-x^2}\right]$

$\Rightarrow \frac{2\cos x}{1-\cos^2 x} = 2\operatorname{cosec} x$

$\Rightarrow \frac{2\cos x}{\sin^2 x} = \frac{2}{\sin x}$

$\Rightarrow \cos x = \sin x$

$\Rightarrow \tan x = 1$

$\therefore x = \frac{\pi}{4}$

Question 14:

Solve $\tan^{-1}\frac{1-x}{1+x} = \frac{1}{2}\tan^{-1} x, (x > 0)$

Solution 14:

$\tan^{-1}\frac{1-x}{1+x} = \frac{1}{2}\tan^{-1} x$

$\Rightarrow \tan^{-1}1-\tan^{-1}x=\frac{1}{2}\tan^{-1}x \qquad \left[\because \quad -\tan^{-1}y=\tan^{-1}\frac{x-y}{1+xy}\right]$

$\Rightarrow \frac{\pi}{4}=\frac{3}{2}\tan^{-1}x$

$\Rightarrow \tan^{-1}x=\frac{\pi}{6}$

$\Rightarrow x=\tan\frac{\pi}{6}$

$\therefore x=\frac{1}{\sqrt{3}}$

Question 15:

Solve $\sin\left(\tan^{-1}x\right), |x|<1$ is equal to

(A) $\frac{x}{\sqrt{1-x^2}}$ **(B)** $\frac{1}{\sqrt{1-x^2}}$

(C) $\frac{1}{\sqrt{1+x^2}}$ **(D)** $\frac{x}{\sqrt{1+x^2}}$

Solution 15:

$\tan y=x \Rightarrow \sin y=\frac{x}{\sqrt{1+x^2}}$

Let $\tan^{-1}x=y$. Then,

$y=\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)\Rightarrow \tan^{-1}x=\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$

$\Rightarrow \sin\left(\tan^{-1}x\right)=\sin\left(\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)\right)=\frac{x}{\sqrt{1+x^2}}$

The correct answer is **D.**

Question 16:

Solve $\sin^{-1}(1-x)-2\sin^{-1}x=\frac{\pi}{2}$, then x is equal to

(A) $0, \frac{1}{2}$ **(B)** $1, \frac{1}{2}$

(C) 0 **(D)** $\frac{1}{2}$

Solution 16:

$\sin^{-1}(1-x)-2\sin^{-1}x=\frac{\pi}{2}$

$\Rightarrow -2\sin^{-1}x=\frac{\pi}{2}-\sin^{-1}(1-x)$

$\Rightarrow -2\sin^{-1}x=\cos^{-1}(1-x)$ ...(1)

Let $\sin^{-1}x=\theta \Rightarrow \sin\theta = x \Rightarrow \cos\theta=\sqrt{1-x^2}$.

$\therefore \theta = \cos^{-1}\left(\sqrt{1-x^2}\right)$

$\therefore \sin^{-1}x=\cos^{-1}\left(\sqrt{1-x^2}\right)$

Therefore, from equation (1), we have

$-2\cos^{-1}\left(\sqrt{1-x^2}\right)=\cos^{-1}(1-x)$

Let, $x=\sin y$. Then, we have:

$-2\cos^{-1}\left(\sqrt{1-\sin^2 y}\right)=\cos^{-1}(1-\sin y)$

$\Rightarrow -2\cos^{-1}(\cos y)=\cos^{-1}(1-\sin y)$

$\Rightarrow -2y=\cos^{-1}(1-\sin y)$

$\Rightarrow 1-\sin y=\cos(-2y)=\cos 2y$

$\Rightarrow 1-\sin y=1-2\sin^2 y$

$\Rightarrow 2\sin^2 y-\sin y=0$

$\Rightarrow \sin y(2\sin y-1)=0$

$\Rightarrow \sin y=0 \ or \ \frac{1}{2}$

$\therefore x=0 \ OR \ x=\frac{1}{2}$

When $x=\frac{1}{2}$, it can be observed that:

L.H.S. $=\sin^{-1}\left(1-\frac{1}{2}\right)-\sin^{-1}\frac{1}{2}$

$=\sin^{-1}\left(\frac{1}{2}\right)-2\sin^{-1}\frac{1}{2}$

$=-\sin^{-1}\frac{1}{2}$

$=\frac{\pi}{6}\neq\frac{\pi}{2}\neq R.H.S.$

$\therefore\ x=\frac{1}{2}$ is not a solution of the given equation.

Thus, $x = 0$.

Hence, the correct answer is **C.**

Question 17:

Solve $\tan^{-1}\left(\frac{x}{y}\right)-\tan^{-1}\frac{x-y}{x+y}$ is equal to

(A) $\frac{\pi}{2}$ **(B)** $\frac{\pi}{3}$

(C) $\frac{\pi}{4}$ **(D)** $\frac{-3\pi}{4}$

Solution 17:

$$\tan^{-1}\left(\frac{x}{y}\right)-\tan^{-1}\left(\frac{x-y}{x+y}\right)$$

$$=\tan^{-1}\left[\frac{\frac{x}{y}-\frac{x-y}{x+y}}{1+\left(\frac{x}{y}\right)\left(\frac{x-y}{x+y}\right)}\right]\qquad\left[\because\quad -\tan^{-1}y=\tan^{-1}\frac{x-y}{1+xy}\right]$$

$$=\tan^{-1}\left[\frac{\frac{x(x+y)-y(x-y)}{y(x+y)}}{\frac{y(x+y)+x(x-y)}{y(x+y)}}\right]$$

$$=\tan^{-1}\left(\frac{x^2+xy-xy+y^2}{xy+y^2+x^2-xy}\right)$$

$$= \tan^{-1}\left(\frac{x^2+y^2}{x^2+y^2}\right) = \tan^{-1} 1 = \frac{\pi}{4}$$

Hence, the correct answer is **C**.