Exercise 5.1

Question 1:
Prove that the function $f(x)=5x-3$ is continuous at $x=0, x=-3$ and at $x=5$.

Solution 1:
The given function is $f(x)=5x-3$
At $x=0$, $f(0)=5\times 0-3=3$
$\lim_{x\to 0} f(x)=\lim_{x\to 0}(5x-3)=5\text{x}0\text{-}3\text{=-}3$
$\therefore \lim_{x\to 0} f(x)=f(0)$
Therefore, f is continuous at $\text{x}=0$
At $x=-3$, $f(-3)=5\text{x}(\text{-}3)-3=-18$
$\lim_{x\to 3} f(x)=\lim_{x\to 3} f(5x-3)=5\text{x}(\text{-}3)-3=-18$
$\therefore \lim_{x\to 3} f(x)=f(-3)$
Therefore, f is continuous at $\text{x}=\text{-}3$
At $x=5$, $f(x)=f(5)=5\text{x}5-3=25-3=22$
$\lim_{x\to 5} f(x)=\lim_{x\to 5}(5x-3)=5\text{x}5\text{-}3=22$
$\therefore \lim_{x\to 5} f(x)=f(5)$
Therefore, f is continuous at $\text{x}=5$

Question 2:
Examine the continuity of the function $f(x)=2x^2-1$ at $x=3$.

Solution 2:
The given function is $f(x)=2x^2-1$
At $x=3$, $f(x)=f(3)=2\text{x}3^2-1=17$
$\lim_{x\to 3} f(x)=\lim_{x\to 3}(2x^2-1)=2\text{x}3^2-1=17$
$\therefore \lim_{x\to 3} f(x)=f=(3)$
Thus, f is continuous, at $x=3$

Question 3:

Examine the following functions for continuity.

a) $f(x)=x-5$

b) $f(x)=\frac{1}{x-5}, x\neq 5$

c) $f(x)=\frac{x^2-25}{x+5}, x\neq 5$

d) $f(x) = |x-5|$

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Solution 3:

a) The given function is $f(x)=x-5$

It is evident that f is defined at every real number k and its value at k is $k-5$.

It is also observed that $\lim\limits_{x\to k} f(x)=\lim\limits_{x\to k} f(x-5)=k=k-5=f(k)$

$\therefore \lim\limits_{x\to k} f(x)=f(k)$

Hence, f is continuous at every real number and therefore, it is a continuous function.

b). The given function is $f(x)=\frac{1}{x-5}, x\neq 5$

for any real number $k\neq 5$, we obtain

$\lim\limits_{x\to k} f(x)=\lim\limits_{x\to k} \frac{1}{x-5}=\frac{1}{k-5}$

Also, $f(k)=\frac{1}{k-5}$ $\quad$ $(\text{As } k\neq 5)$

$\therefore \lim\limits_{x\to k} f(x)=f(k)$

Hence, f is continuous at every point in the domain of f and therefore, it is a continuous function.

c). The given function is $f(x)=\frac{x^2-25}{x+5}, x\neq -5$

For any real number $c\neq -5$, we obtain

$\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c} \frac{x^2-25}{x+5}=\lim\limits_{x\to c} \frac{(x+5)(x-5)}{x+5}=\lim\limits_{x\to c}(x-5)=(c-5)$

Also, $f(c)=\frac{(c+5)(c-5)}{c+5}=c(c-5)(\text{as } c\neq 5)$

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

Hence f is continuous at every point in the domain of f and therefore. It is continuous function.

d). The given function is $f(x)=|x-5|=\begin{cases}5-x, if\ x<5\\ x-5, if\ x\geq 5\end{cases}$

This function f is defined at all points of the real line.

Let c be a point on a real line. Then, $c<5$ or $c=5$ or $c>5$

case $I : c<5$

Then, $f(c)=5-c$

$$\lim_{x\to c} f(x)=\lim_{x\to c}(5-x)=5-c$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous at all real numbers less than 5.

case $II : c=5$

Then, $f(c)=f(5)=(5-5)=0$

$$\lim_{x\to 5^-} f(x)=\lim_{x\to 5}(5-x)=(5-5)=0$$

$$\lim_{x\to 5^+} f(x)=\lim_{x\to 5}(x-5)=0$$

$$\therefore \lim_{x\to c^-} f(x)=\lim_{x\to c^+} f(x)=f(c)$$

Therefore, f is continuous at $x=5$

case $III : c>5$

Then, $f(c)=f(5)=c-5$

$$\lim_{x\to c} f(x)=\lim_{x\to c} f(x-5)=c-5$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous at real numbers greater than 5.

Hence, f is continuous at every real number and therefore, it is a continuous function.

Question 4:

Prove that the function $f(x)=x^n$ is continuous at $x=n$ is a positive integer.

Solution 4:

The given function is $f(x)=x^n$

It is evident that f is defined at all positive integers, n, and its value at n is n^n.

Then, $\lim\limits_{x\to n} f(n) = \lim\limits_{x\to n} f(x^n) = n^n$

$\therefore \lim\limits_{x\to n} f(x) = f(n)$

Therefore, f is continuous at n, where n is a positive integer.

Question 5:

Is the function f defined by $f(x) = \begin{cases} x, if\ x \le 1 \\ 5, if\ x > 1 \end{cases}$

Continuous at $x = 0$? At $x = 1$?, At $x = 2$?

Solution 5:

The given function f is $f(x) = \begin{cases} x, if\ x \le 1 \\ 5, if\ x > 1 \end{cases}$

At x = 0,

It is evident that f is defined at 0 and its value at 0 is 0.

Then, $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} x = 0$

$\therefore \lim\limits_{x\to 0} f(x) = f(0)$

Therefore, f is continuous at $x = 0$

At $x = 1$,

f is defined at 1 and its value at is 1.

The left hand limit of f at $x = 1$ is,

$\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} x = 1$

The right hand limit of f at $x = 1$ is,

$\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} f(5)$

$\therefore \lim\limits_{x\to 1^-} f(x) \ne \lim\limits_{x\to 1^+} f(x)$

Therefore, f is not continuous at $x = 1$

At $x = 2$,

f is defined at 2 and its value at 2 is 5.

Then, $\lim\limits_{x\to 2} f(x) = \lim\limits_{x\to 2} f(5) = 5$

$\therefore \lim\limits_{x\to 2} f(x) = f(2)$

Therefore, f is continuous at $x=2$

Question 6:

Find all points of discontinuous of f, where f is defined by

$$f(x)=\begin{cases}2x+3, & if\ x\le 2\\ 2x-3, & if\ x>2\end{cases}$$

Solution 6:

The give function f is $f(x)=\begin{cases}2x+3, & if\ x\le 2\\ 2x-3, & if\ x>2\end{cases}$

It is evident that the given function f is defined at all the points of the real line.

Let c be a point on the real line. Then, three cases arise.

I. $c<2$
II. $c>2$
III. $c=2$

$Case\ (i)\ c<2$

Then, $f(x)=2x+3$

$$\lim_{x\to c} f(x)=\lim_{x\to c}(2x+3)=2c+3$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous at all points, x, such that $x<2$

$Case\ (ii)\ c>2$

Then, $f(c)=2c-3$

$$\lim_{x\to c} f(x)=\lim_{x\to c}(2x-3)=2c-3$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous at all points x, such that $x>2$

$Case\ (iii)\ c=2$

Then, the left hand limit of f at $x=2$ is,

$$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-}(2x+3)=2\text{x}2+3=7$$

The right hand limit of f at $x=2$ is,

$\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+}(2x+3) = 2\text{x}2-3=1$

It is observed that the left and right hand limit of f at $x=2$ do not coincide.

Therefore, f is not continuous at $x=2$

Hence, $x=2$ is the only point of discontinuity of f.

Question 7:

Find all points of discontinuity of f, where f is defined by

$$f(x)=\begin{cases} |x|+3, & if\ \ x\le -3 \\ -2x, & if\ \ -3<x<3 \\ 6x+2, & if\ \ x\ge 3 \end{cases}$$

Solution 7:

The given function f is $f(x)=\begin{cases} |x|+3, & if\ \ x\le -3 \\ -2x, & if\ \ -3<x<3 \\ 6x+2, & if\ \ x\ge 3 \end{cases}$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I :

If $c<-3$, then $f(c)=-c+3$

$\lim\limits_{x\to c} f(x) = \lim\limits_{x\to c}(-x+3) = -c+3$

$\therefore \lim\limits_{x\to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x<-3$

Case II :

If $c=-3$, then $f(-3)=-(-3)+3=6$

$\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^-}(-x+3) = -(-3)+3=6$

$\therefore \lim\limits_{x\to 3^+} f(x) = \lim\limits_{x\to 3^+} f(-2x) = 2\text{x}(-3)=6$

$\therefore \lim\limits_{x\to 3} f(x) = f(-3)$

Therefore, f is continuous at $x=-3$

Case III :

If $-3<c<3$, then $f(c)=-2c$ and $\lim\limits_{x\to c} f(x)=\lim\limits_{x\to 3c}(-2x)=-2c$

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

Therefore, f is continuous in $(-3,3)$.

Case IV :

If $c=3$, then the left hand limit of f at $x=3$ is,

$\lim\limits_{x\to 3^-} f(x)=\lim\limits_{x\to 3^-} f(-2x)=-2x3=-6$

The right hand limit of f at $x=3$ is,

$\lim\limits_{x\to 3^+} f(x)=\lim\limits_{x\to 3^+} f(6x+2)=6x3+2=20$

It is observed that the left and right hand limit of f at $x=3$ do not coincide.

Therefore, f is not continuous at $x=3$

Case V :

If $c>3$, then $f(c)=6c+2$ and $\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}(6x+2)=6c+2$

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

Therefore, f is continuous at all points x, such that $x>3$

Hence, $x=3$ is the only point of discontinuity of f.

Question 8:

Find all points of discontinuity of f, where f is defined by $f(x)=\begin{cases}\frac{|x|}{x}, & if\ \ x\neq 0\\ 0, & if\ \ x=0\end{cases}$

Solution 8:

The given function f is $f(x)=\begin{cases}\frac{|x|}{x}, & if\ \ x\neq 0\\ 0, & if\ \ x=0\end{cases}$

It is known that, $x<0\Rightarrow |x|=-x$ and $x>0\Rightarrow |x|=x$

Therefore, the given function can be rewritten as

$$f(x)=\begin{cases}\frac{|x|}{x}=\frac{-x}{x}=-1 \; if \;\; x<0\\ 0, \; if \;\; x=0\\ \frac{|x|}{x}=\frac{x}{x}=1 \; if \;\; x>0\end{cases}$$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I :

If $c<0$, then $f(c)=-1$

$$\lim_{x\to c} f(x)=\lim_{x\to c}(-1)=-1$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous at all points $x<0$

Case II :

If $c=0$, then the left hand limit of f at $x=0$ is,

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}(-1)=-1$$

The right hand limit of f at $x=0$ is,

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}(1)=1$$

It is observed that the left and right hand limit of f at $x=0$ do not coincide.

Therefore, f is not continuous at $x=0$

Case III :

If $c>0$, $f(c)=1$

$$\lim_{x\to c} f(x)=\lim_{x\to c}(1)=1$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous at all points x, such that $x>0$

Hence, $x=0$ is the only point of discontinuity of f .

Question 9:

Find all points of discontinuity of f , where f is defined by $f(x)=\begin{cases}\frac{x}{|x|}, \; if \;\; x<0\\ -1, if \;\; x\geq 0\end{cases}$

Solution 9:

The given function f is $f(x)=\begin{cases}\dfrac{x}{|x|}, & if\ \ x<0\\ -1, if & x\ge 0\end{cases}$

It is known that, $x<0\Rightarrow |x|=-x$

Therefore, the given function can be rewritten as

$$f(x)=\begin{cases}\dfrac{x}{|x|}, & if\ \ x<0\\ -1, if & x\ge 0\end{cases}$$

$\Rightarrow f(x)=-1$ for all $x\in \mathbf{R}$

Let c be any real number. Then , $\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}(-1)=-1$

Also, $f(c)=-1=\lim\limits_{x\to c} f(x)$

Therefore, the given function is continuous function.
Hence, the given function has no point of discontinuity.

Question 10:

Find all the points of discontinuity of f , where f is defined by $f(x)=\begin{cases}x+1 & if\ \ x\ge 1\\ x^2+1, & f\ \ x<1\end{cases}$

Solution 10:

The given function f is $f(x)=\begin{cases}x+1 & if\ \ x\ge 1\\ x^2+1, & f\ \ x<1\end{cases}$

The given function f is defined at all the points of the real line.
Let c be a point on the real line.
Case I :

If $c<1$ then $f(c)=c^2+1$ and $\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c} f(x^2+1)=c^2+1$

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

Therefore , f is continuous at all points x , such that $x<1$
Case II :

If $c=1$, then $f(c)=f(1)=1+1=2$

The left hand limit of f at $x=1$ is,

$$\lim_{x\to1^-} f(x) = \lim_{x\to1^-}\left(x^2+1\right) = 1^2+1 = 2$$

The right hand limit of f at $x=1$ is,

$$\lim_{x\to1^+} f(x) = \lim_{x\to1^+}\left(x^2+1\right) = 1^2+1 = 2$$

$$\therefore \lim_{x\to1} f(x) = f(c)$$

Therefore, f is continuous at $x=1$

Case III :

If $c>1$, then $f(c)=c+1$

$$\lim_{x\to c} f(x) = \lim_{x\to c}(x+1) = c+1$$

$$\therefore \lim_{x\to c} f(x) = f(c)$$

Therefore, f is continuous at all points x, such that $x>1$

Hence, the given function f has no points of discontinuity.

Question 11:

Find all points of discontinuity of f, where f is defined by $f(x)=\begin{cases} x^3-3, & if \;\; x\le 2 \\ x^2+1, & if \;\; x>2 \end{cases}$

Solution 11:

The given function f is $f(x)=\begin{cases} x^3-3, & if \;\; x\le 2 \\ x^2+1, & if \;\; x>2 \end{cases}$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I :

If $c<2$, then $f(c)=c^3-3$ and $\lim_{x\to c} f(x) = \lim_{x\to c}\left(x^3-3\right) = c^3-3$

$$\therefore \lim_{x\to c} f(x) = f(c)$$

Therefore, f is continuous at all points x, such that $x<2$

Case II :

If $c=2$, then $f(c)=f(2)=2^3-3=5$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}\left(x^3 - 3\right) = 2^3 - 3 = 5$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}\left(x^2 + 1\right) = 2^2 + 1 = 5$$

$$\therefore \lim_{x\to 2} 1 f(x) = f(2)$$

Therefore, f is continuous at $x = 2$

Case III :

If $c > 2$, then $f(c) = c^2 + 1$

$$\lim_{x\to c} f(x) = \lim_{x\to c}\left(x^2 + 1\right) = c^2 + 1$$

$$\therefore \lim_{x\to c} f(x) = f(c)$$

Therefore, f is continuous at all points x, such that $x > 2$

Thus, the given function f is continuous at every point on the real line.

Hence, f has no point of discontinuity.

Question 12:

Find all points of discontinuity of f, where f is defined by $f(x) = \begin{cases} x^{10} - 1, & if\ \ x \le 1 \\ x^2, & if\ \ x > 1 \end{cases}$

Solution 12:

The given function f is $f(x) = \begin{cases} x^{10} - 1, & if\ \ x \le 1 \\ x^2, & if\ \ x > 1 \end{cases}$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I :

If $c < 1$, then $f(c) = c^{10} - 1$ and $\lim_{x\to c} f(x) = \lim_{x\to c}\left(x^{10} - 1\right) = c^{10} - 1$

$$\therefore \lim_{x\to c} f(x) = f(c)$$

Therefore, f is continuous at all points x, such that $x < 1$

Case II :

If $c = 1$, then the left hand limit of f at $x = 1$ is,

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}\left(x^{10} - 1\right) = 10^{10} - 1 = 1 - 1 = 0$$

The right hand limit of f at $x = 1$ is,

$\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+}(x^2) = 1^2 = 1$

It is observed that the left and right hand limit of f at $x=1$ do not coincide.

Therefore, f is not continuous at $x=1$

Case III :

If $c>1$, then $f(c)=c^2$

$\lim\limits_{x\to c} f(x) = \lim\limits_{x\to c}(x^2) = c^2$

$\therefore \lim\limits_{x\to c} f(x) = f(c)$

Therefore, f is continuous at all pints x, such that $x>1$

Thus, from the above observation, it can be concluded that $x=1$ is the only point of discontinuity of f.

Question 13:

Is the function defined by $f(x)=\begin{cases} x+5, & if\ x\le 1 \\ x-5, & if\ x>1 \end{cases}$ a continuous function?

Solution 13:

The given function is $f(x)=\begin{cases} x+5, & if\ x\le 1 \\ x-5, & if\ x>1 \end{cases}$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I :

If $c<1$, then $f(c)=c+5$ and $\lim\limits_{x\to c} f(x) = \lim\limits_{x\to c}(x+5) = c+5$

$\therefore \lim\limits_{x\to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x<1$

Case II :

If $c=1$, then $f(1)=1+5=6$

The left hand limit of f at $x=1$ is,

$\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-}(x+5) = 1+5 = 6$

The right hand limit of f at $x=1$ is,

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+}(x-5) = 1-5 = -4$$

It is observed that the left and right hand limit of f at $x=1$ do not coincide.

Therefore, f is not continuous at $x=1$

Case III :

If $c>1$, then $f(c)=c-5$ and $\lim_{x \to c} f(x) = \lim_{x \to c}(x-5) = c-5$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x>1$

Thus, from the above observation, it can be concluded that $x=1$ is the only point of discontinuity of f.

Question 14:

Discuss the continuity of the function f, where f is defined by $f(x) = \begin{cases} 3, \text{ if } 0 \le x \le 1 \\ 4, \text{ if } 1 < x < 3 \\ 5, \text{ if } 3 \le x \le 10 \end{cases}$

Solution 14:

The given function is $f(x) = \begin{cases} 3, \text{ if } 0 \le x \le 1 \\ 4, \text{ if } 1 < x < 3 \\ 5, \text{ if } 3 \le x \le 10 \end{cases}$

The given function is defined at all points of the interval $[0,10]$.

Let c be a point in the interval $[0,10]$.

Case I :

If $0 \le c < 1$, then $f(c)=3$ and $\lim_{x \to c} f(x) = \lim_{x \to c}(3) = 3$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous in the interval $[0,1)$.

Case II :

If $c=1$, then $f(3)=3$

The left hand limit of f at $x=1$ is,

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-}(3) = 3$

The right hand limit of f at $x=1$ is ,

$\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+}(4) = 4$

It is observed that the left and right hands limit of f at $x=1$ do not coincide.

Therefore, f is not continuous at $x=1$

Case III :

If $1<c<3$, then $f(c)=4$ and $\lim\limits_{x\to c} f(x) = \lim\limits_{x\to c}(4) = 4$

$\therefore \lim\limits_{x\to c} f(x) = f(c)$

Therefore, f is continuous at all points of the interval $(1,3)$.

Case IV :

If $c=3$, then $f(c)=5$

The left hand limit of f at $x=3$ is,

$\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^-}(4) = 4$

The right hand limit of f at $x=3$ is,

$\lim\limits_{x\to 3^+} f(x) = \lim\limits_{x\to 3^+}(5) = 5$

It is observed that the left and right hand limit of f at $x=3$ do not coincide.

Therefore, f is not continuous at $x=3$

Case V :

If $3<c\le 10$, then $f(c)=5$ and $\lim\limits_{x\to c} f(x) = \lim\limits_{x\to c}(5) = 5$

$\therefore \lim\limits_{x\to c} f(x) = f(c)$

Therefore, f is continuous at all points of the interval $(3,10]$.

Hence , f is not continuous at $x=1$ and $x=3$.

Question 15:

Discuss that continuity of the function f, where f is defined by $f(x)=\begin{cases} 2x, & if\ x<0 \\ 0, & if\ 0\le x\le 1 \\ 4x, & if\ x>1 \end{cases}$

Solution 15:

The given function is $f(x)=\begin{cases}2x, & if\ x<0\\ 0, & if\ 0\le x\le 1\\ 4x, & if\ x>1\end{cases}$

The given function is defined at all points of the real line.

Let c be a point on the real line.

Case I :

If $c<0$, then $f(c)=2c$

$\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}(2x)=2c$

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

Therefore, f is continuous at all points x, such that $x<0$

Case II :

If $c=0$, then $f(c)=f(0)=0$

The left hand limit of f at $x=0$ is,

$\lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0^-}(2x)=2\text{ x }0=0$

The right hand limit of f at $x=0$ is,

$\lim\limits_{x\to 0^+} f(x)=\lim\limits_{x\to 0^+}(0)=0$

$\therefore \lim\limits_{x\to 0} f(x)=f(0)$

Therefore, f is continuous at $x=0$

Case III :

If $0<c<1$, then $f(x)=0$ and $\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}(0)=0$

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

Therefore, f is continuous at all points of the interval $(0,1)$.

Case IV :

If $c=1$, then $f(c)=f(1)=0$

The left hand limit of f at $x=1$ is ,

$\lim\limits_{x\to 1^-} f(x)=\lim\limits_{x\to 1^-}(0)=0$

The right hand limit of f at $x=1$ is,

$\lim\limits_{x\to 1^+} f(x)=\lim\limits_{x\to 1^+}(4x)=4\text{ x }1=4$

It is observed that the left and right hand limits of f at $x=1$ do not coincide.

Therefore, f is not continuous at $x=1$

Case V:

If $c<1$, then $f(c)=4c$ and $\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}(4x)=4c$

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

Therefore, f is continuous at all points x, such that $x>1$

Hence, f is not continuous only at $x=1$

Question 16:

Discuss the continuity of the function f, where f is defined by $f(x)=\begin{cases}-2, & if\ x\le -1\\ 2x, & if\ -1<x\le 1\\ 2, & if\ x>1\end{cases}$

Solution 16:

The given function f is $f(x)=\begin{cases}-2, & if\ x\le -1\\ 2x, & if\ -1<x\le 1\\ 2, & if\ x>1\end{cases}$

The given function is defined at all points of the real line.
Let c be a point on the real line.

Case I:

If $c<-1$, then $f(c)=-2$ and $\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}(-2)=-2$

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

Therefore, f is continuous at all points x, such that $x<-1$

Case II:

If $c=-1$, then $f(c)=f(-1)=-2$

The left hand limit of f at $x=-1$ is,

$\lim\limits_{x\to 1^-} f(x)=\lim\limits_{x\to 1^-}(-2)=-2$

The right hand limit of f at $x=-1$ is,

$\lim\limits_{x\to -1^+} f(x)=\lim\limits_{x\to -1^+}=2\text{ x}(-1)=-2$

$\therefore \lim\limits_{x\to -1} f(x)=f(-1)$

Therefore, f is continuous at $x=-1$

Case III :

If $-1<c<1$, then $f(c)=2c$

$$\lim_{x \to c} f(x)=\lim_{x \to c}(2x)=2c$$

$$\therefore \lim_{x \to c} f(x)=f(c)$$

Therefore, f is continuous at all points of the interval $(-1,1)$.

Case IV :

If $c=1$, then $f(c)=f(1)=2\text{x}1=2$

The left hand limit of f at $x=1$ is,

$$\lim_{x \to 1^-} f(x)=\lim_{x \to 1^-}(2x)=2\text{x}1=2$$

The right hand limit of f at $x=1$ is,

$$\lim_{x \to 1^+} f(x)=\lim_{x \to 1^+} 2=2$$

$$\therefore \lim_{x \to 1} f(x)=f(c)$$

Therefore, f is continuous at $x=2$

Case V :

If $c>1$, $f(c)=2$ and $\lim_{x \to 2} f(x)=\lim_{x \to 2}(2)=2$

$$\therefore \lim_{x \to c} f(x)=f(c)$$

Therefore, f is continuous at all points, x, such that $x>1$

Thus, from the above observations, it can be concluded that f is continuous at all points of the real line.

Question 17:

Find the relationship be a and b so that the function f defined by $f(x)=\begin{cases} ax+1, \text{ if } x \le 3 \\ bx+3, \text{ if } x>3 \end{cases}$ is continuous at $x=3$.

Solution 17:

The given function f is $f(x)=\begin{cases} ax+1, \text{ if } x \le 3 \\ bx+3, \text{ if } x>3 \end{cases}$

If f is continuous at $x=3$, then

$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} f(x) = f(3)$ .....(1)

Also,

$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} f(ax+1) = 3a+1$

$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} f(bx+1) = 3b+3$

$f(3) = 3a+1$

Therefore, from (1), we obtain

$3a+1 = 3b+3 = 3a+1$

$\Rightarrow 3a+1 = 3b+3$

$\Rightarrow 3a = 3b+2$

$\Rightarrow a = b+\frac{2}{3}$

Therefore, the required relationship is given by , $a = b+\frac{2}{3}$

Question 18:

For what value of λ is the function defined by $f(x) = \begin{cases} \lambda(x^2-2x), if \ \ x \le 0 \\ 4x+1, \quad if \ \ x > 0 \end{cases}$ continuous at $x=0$? what about continuity at $x=1$?

Solution 18:

The given function f is $f(x) = \begin{cases} \lambda(x^2-2x), if \ \ x \le 0 \\ 4x+1, \quad if \ \ x > 0 \end{cases}$

If f is continuous at $x=0$, then

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} f(x) = f(0)$

$\Rightarrow \lim_{x \to 0^-} \lambda(x^2-2x) = \lim_{x \to 0^+} (4x+1) = \lambda(0^2 - 2 \text{ x} 0)$

$\Rightarrow \lambda(0^2 - 2 \text{ x} 0) = 4 \text{ x} 0 + 1 = 0$

$\Rightarrow 0=1=0,$ which is not possible

Therefore, there is no value of λ for which f is continuous at $x=0$

At $x=1$,

$f(1) = 4x+1 = 4 \text{ x } 1+1 = 5$

$\lim_{x \to 1} (4x+1) = 4\text{x}1+1=5$

$\therefore \lim_{x\to 1} f(x) = f(1)$

Therefore, for any values of λ, f is continuous at $x=1$

Question 19:

Show that the function defined by $g(x)=x-[x]$ is discontinuous at all integral point.

Here $[x]$ denotes the greatest integer less than or equal to x.

Solution 19:

The given function is $g(x)=x-[x]$

It is evident that g is defined at all integral points.

Let n be a integer.

Then,

$g(n)=n-[n]=n-n=0$

The left hand limit of f at $x=n$ is,

$\lim_{x\to n^-} g(x)=\lim_{x\to n^-}[x-[x]]=\lim_{x\to n^-}(x)-\lim_{x\to n^-}[x]=n-(n-1)=1$

The right hand limit of f at $x=n$ is,

$\lim_{x\to n^+} g(x)=\lim_{x\to n^+}[x-[x]]=\lim_{x\to n^+}(x)-\lim_{x\to n^+}[x]=n-n=0$

It is observed that the left and right hand limits of f at $x=n$ do not coincide.

Therefore, f is not continuous at $x=n$

Hence, g is discontinuous at all integral points.

Question 20:

Is the function defined by $f(x)=x^2-\sin x+5$ continuous at $x=\pi$?

Solution 20:

The given function is $f(x)=x^2-\sin x+5$

It is evident that f id defined at $x=\pi$

At $x=\pi$, $f(x)=f(\pi)=\pi^2-\sin\pi+5=\pi^2-0+5=\pi^2+5$

Consider $\lim_{x\to\pi} f(x)=\lim_{x\to\pi}(x^2-\sin x+5)$

Put $x=\pi+h$

If $x \to \pi$, then it is evident that $h \to 0$

$$\therefore \lim_{x\to\pi} f(x) = \lim_{x\to\pi}\left(x^2 - \sin x\right) + 5)$$

$$= \lim_{h\to 0}\left[(\pi+h)^2 - \sin(\pi+h) + 5\right]$$

$$= \lim_{h\to 0}(\pi+h)^2 - \lim_{h\to 0}\sin(\pi+h) + \lim_{h\to 0}5$$

$$= (\pi+0)^2 - \lim_{h\to 0}[\sin\pi\cosh + \cos\pi + \sinh] + 5$$

$$= \pi^2 - \lim_{h\to 0}\sin\pi\cosh - \lim_{h\to 0}\cos\pi\sinh + 5$$

$$= \pi^2 - \sin\pi\cos 0 - \cos\pi\sin 0 + 5$$

$$= \pi^2 - 0 \text{ x } 1\text{-}(\text{-}1)\text{x}0\text{+}5$$

$$= \pi^2 + 5$$

$$\therefore \lim_{x\to x} f(x) = f(\pi)$$

Therefore, the given function f is continuous at $x = \pi$

Question 21:

Discuss the continuity of the following functions.

a) $f(x) = \sin x + \cos x$

b) $f(x) = \sin x - \cos x$

c) $f(x) = \sin x \text{ x } \cos x$

Solution 21:

It is known that if g and h are two continuous functions, then $\mathrm{g} + \mathrm{h}$, $\mathrm{g} - \mathrm{h}$ and $g \cdot h$ are also continuous.

It has to proved first that $\mathrm{g}(\mathrm{x}) = \sin x$ and $\mathrm{h}(\mathrm{x}) = \cos x$ are continuous functions.

Let $\mathrm{g}(\mathrm{x}) = \sin x$

It is evident that $\mathrm{g}(\mathrm{x}) = \sin x$ is defined for every real number.

Let c be a real number. Put $\mathrm{x} = c + h$

If $x \to c$, then $h \to 0$

$\mathrm{g}(\mathrm{c}) = \sin c$

$$\lim_{x\to c} g(x) = \lim_{x\to c} g \sin x$$

$$= \lim_{h\to 0}\sin(c+h)$$

$$= \lim_{h \to 0} [\sin c \cos h + \cos c \sin h]$$
$$= \lim_{h \to 0} (\sin c \cos h) + \lim_{h \to 0} (\cos c \sin h)$$
$$= \sin c \cos 0 + \cos c \sin 0$$
$$= \sin c + 0$$
$$= \sin c$$

$\therefore \lim_{x \to c} g(x) = g(c)$

Therefore , g is a continuous function.

Let $h(x) = \cos x$

It is evident that $h(x) = \cos x$ is defined for every real number.

Let c be a real number. Put $x = c + h$

If $x \to c$, then $h \to 0$

$h(c) = \cos c$

$$\lim_{x \to c} h(x) = \lim_{x \to c} \cos x$$
$$= \lim_{h \to 0} \cos(c + h)$$
$$= \lim_{h \to 0} [\cos c \cos h - \sin c \sin h]$$
$$= \lim_{h \to 0} \cos c \cos h - \lim_{h \to 0} \sin c \sin h$$
$$= \cos c \cos 0 - \sin c \sin 0$$
$$= \cos c \text{ x } 1 - \sin c \text{ x } 0$$
$$= \cos c$$

$\therefore \lim_{h \to 0} h(x) = h(c)$

Therefore, h is a continuous function.

Therefore, it can be concluded that

a) $f(x) = g(x) + h(x) = \sin x + \cos x$ is a continuous function

b) $f(x) = g(x) - h(x) = \sin x - \cos x$ is a continuous function

c) $f(x) = g(x)$ x $h(x) = \sin x$ x $\cos x$ is a continuous function

Question 22:

Discuss the continuity of the cosine, cosecant, secant and cotangent functions.

Solution 22:

It is known that if g and h are two continuous functions, then

i. $\frac{h(x)}{g(x)}, g(x) \neq 0 \text{ is continuous}$

ii. $\frac{1}{g(x)}, g(x) \neq 0 \text{ is continuous}$

iii. $\frac{1}{h(x)}, h(x) \neq 0 \text{ is continuous}$

It has to be proved first that $g(x) = \sin x$ and $h(x) = \cos x$ are continuous functions.

Let $g(x) = \sin x$

It is evident that $g(x) = \sin x$ is defined for every real number.

Let c be a real number. Put $x = c + h$
If $x \to c$, then $h \to 0$

$g(c) = \sin x$

$$\begin{aligned}
\lim_{x \to c} g(c) &= \lim_{x \to c} \sin x \\
&= \lim_{h \to 0} \sin(c + h) \\
&= \lim_{h \to 0} [\sin c \cosh + \cos c \sinh] \\
&= \lim_{h \to 0} (\sin c \cosh) + \lim_{h \to 0} (\cos c \sinh) \\
&= \sin c \cos 0 + \cos c \sin 0 \\
&= \sin c + 0 \\
&= \sin c
\end{aligned}$$

$\therefore \lim_{x \to c} g(x) = g(c)$

Therefore, g is a continuous function.

Let $h(x) = \cos x$

It is evident that $h(x) = \cos x$ is defined for every real number.

Let c be a real number. Put $x = c + h$
If $x \to c$, then $h \to 0$ x

$h(c) = \cos c$

$\lim_{x \to c} h(x) = \lim_{x \to c} \cos x$

$$= \lim_{h \to 0} \cos(c+h)$$
$$= \lim_{h \to 0} [\cos c \cosh - \sin c \sinh]$$
$$= \lim_{h \to 0} \cos c \cosh - \lim_{h \to 0} \sin c \sinh$$
$$= \cos c \cos 0 - \sin c \sin 0$$
$$= \cos c \text{ x } 1 \text{ - sinc x } 0$$
$$= \cos c$$
$$\therefore \lim_{x \to c} h(x) = h(c)$$

Therefore, $h(x) = \cos x$ is continuous function.

It can be concluded that,

$$\cos ec\, x = \frac{1}{\sin x}, \sin x \neq 0 \textit{ is continuous}$$
$$\Rightarrow \cos ec\, x, x \neq n\pi (n \in Z) \textit{is continuous}$$

Therefore, secant is continuous except at $X = np, n \hat{I} Z$

$$\sec x = \frac{1}{\cos x}, \quad \cos x \neq 0 \quad \textit{is continuous}$$
$$\Rightarrow \sec x, x \neq (2n+1)\frac{\pi}{2}(n \in Z) \quad \textit{is continuous}$$

Therefore, secant is continuous except at $x = (2n+1)\frac{\pi}{2}(n \in Z)$

$$\cot x = \frac{\cos x}{\sin x}, \quad \sin x \neq 0 \quad \textit{is continuous}$$
$$\Rightarrow \cot x, x \neq n\pi (n \in Z) \quad \textit{is continuous}$$

Therefore, cotangent is continuous except at $x = np, n\hat{I}Z$

Question 23:

Find the points of discontinuity of f, where $f(x) = \begin{cases} \frac{\sin x}{x}, & if\ x < 0 \\ x+1, & if\ x \geq 0 \end{cases}$

Solution 23:

The given function f is $f(x) = \begin{cases} \frac{\sin x}{x}, & if\ x < 0 \\ x+1, & if\ x \geq 0 \end{cases}$

It is evident that f is defined at all points of the real line.

Let c be a real number.

Case I :

If $c<0$, then $f(c)=\frac{\sin c}{c}$ and $\lim_{x\to c} f(x)=\lim_{x\to c}\left(\frac{\sin x}{x}\right)=\frac{\sin c}{c}$

$\therefore \lim_{x\to c} f(x)=f(c)$

Therefore, f is continuous at all points x, such that $x<0$

Case II :

If $c>0$, then $f(c)=c+1$ and $\lim_{x\to c} f(x)=\lim_{x\to c}(x+1)=c+1$

$\therefore \lim_{x\to c} f(x)=f(c)$

Therefore, f is continuous at all points x, such that $x>0$

Case III :

If $c=0$, then $f(c)=f(0)=0+1=1$

The left hand limit of f at $x=0$ is,

$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}\frac{\sin x}{x}=1$

The right hand limit of f at $x=0$ is,

$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}(x+1)=1$

$\therefore \lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x)=f(0)$

Therefore, f is continuous at $x=0$

From the above observations, it can be conducted that f is continuous at all points of the real line.

Thus, f has no point of discontinuity.

Question 24:

Determine if f defined by $f(x)=\begin{cases} x^2\sin\frac{1}{x}, & if \ \neq 0 \\ 0, & if \ x=0 \end{cases}$ is a continuous function?

Solution 24:

The given function f is $f(x)=\begin{cases} x^2\sin\frac{1}{x}, & if \neq 0 \\ 0, & if\ x=0 \end{cases}$

It is evident that f is defined at all points of the real line.

Let c be a real number.

Case I :

If $c\neq 0$, then $f(c)=c^2\sin\frac{1}{c}$

$$\lim_{x\to c} f(x)=\lim_{x\to c}\left(x^2\sin\frac{1}{x}\right)=\left(\lim_{x\to c} x^2\right)\left(\lim_{x\to c}\sin\frac{1}{x}\right)=c^2\sin\frac{1}{c}$$

$\therefore \lim_{x\to c} f(x)=f(c)$

Therefore, f is continuous at all points $x\neq 0$

Case II :

If $c=0$, then $f(0)=0$

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}\left(x^2\sin\frac{1}{x}\right)=\lim_{x\to 0}\left(x^2\sin\frac{1}{2}\right)$$

It is known that, $-1\leq\sin\frac{1}{x}\leq 1,\ \ x\neq 0$

$\Rightarrow -x^2\leq\sin\frac{1}{x}\leq x^2$

$\Rightarrow \lim_{x\to 0}\left(-x^2\right)\leq\lim_{x\to 0}\left(x^2\sin\frac{1}{x}\right)\leq\lim_{x\to 0} x^2$

$\Rightarrow 0\leq\lim_{x\to 0}\left(x^2\sin\frac{1}{x}\right)\leq 0$

$\Rightarrow \lim_{x\to 0}\left(x^2\sin\frac{1}{x}\right)=0$

$\therefore \lim_{x\to 0^-} f(x)=0$

Similarly, $\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}\left(x^2\sin\frac{1}{x}\right)=\lim_{x\to 0}\left(x^2\sin\frac{1}{x}\right)=0$

$\therefore \lim_{x\to 0^-} f(x)=f(0)=\lim_{x\to 0^+} f(x)$

Therefore, f is continuous at $x=0$

From the above observations, it can be concluded that f is continuous at every point of the real line.

Thus, f is a continuous function.

Question 25:

Examine the continuity of f, where f is defined by $f(x)=\begin{cases}\sin x-\cos x, if & x\neq 0\\ 1 & if \ \ x=0\end{cases}$

Solution 25:

The given function f is $f(x)=\begin{cases}\sin x-\cos x, if & x\neq 0\\ 1 & if \ \ x=0\end{cases}$

It is evident that f is defined at all points of the real line.

Let c be a real number.

Case I :

If $c\neq 0$, then $f(c)=\sin c-\cos c$

$$\lim_{x\to c} f(x)=\lim_{x\to c}(\sin x-\cos x)=\sin c-\cos c$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous at al points x, such that $x\neq 0$

Case II :

If $c=0$, then $f(0)=-1$

$$\lim_{x\to o^-} f(x)=\lim_{x\to o}(\sin x-\cos x)=\sin 0-\cos 0=0-1=-1$$

$$\lim_{x\to o^+} f(x)=\lim_{x\to o}(\sin x-\cos x)=\sin 0-\cos 0=0-1=-1$$

$$\therefore \lim_{x\to o^-} f(x)=\lim_{x\to o^+} f(x)=f(0)$$

Therefore, f is continuous at $x=0$

From the above observations, it can be concluded that f is continuous at every point of the real line.

Thus, f is a continuous function.

Question 26:

Find the values of k so that the function f is continuous at the indicated point.

$$f(x)\begin{cases}\dfrac{k\cos x}{\pi-2x}, & if \ \ x\neq\dfrac{\pi}{2}\\ 3, & if \ \ x=\dfrac{\pi}{2}\end{cases} \qquad atx=\frac{\pi}{2}$$

Solution 26:

The given function f is $f(x)\begin{cases} \dfrac{k\cos x}{\pi-2x}, & if \quad x \neq \dfrac{\pi}{2} \\ 3, & if \quad x=\dfrac{\pi}{2} \end{cases}$

The given function f is continuous at $x=\dfrac{\pi}{2}$, it is defined at $x=\dfrac{\pi}{2}$ and if the value of the f at $x=\dfrac{\pi}{2}$ equals the limit of f at $x=\dfrac{\pi}{2}$.

It is evident that f is defined at $x=\dfrac{\pi}{2}$ and $f\left(\dfrac{\pi}{2}\right)=3$

$$\lim_{x\to}\frac{\pi}{2} f(x)=\lim_{x\to\frac{\pi}{2}}\frac{k\cos x}{\pi-2x}$$

Put $x=\dfrac{\pi}{2}+h$

Then, $x\to\dfrac{\pi}{2}\Rightarrow h\to 0$

$$\therefore \lim_{x\to\frac{\pi}{2}} f(x)=\lim_{x\to\frac{\pi}{2}}\frac{k\cos x}{\pi-2x}=\lim_{h\to 0}\frac{k\cos\left(\frac{\pi}{2}+h\right)}{\pi-2\left(\frac{\pi}{2}+h\right)}$$

$$=k\lim_{h\to 0}\frac{-\sinh}{-2h}=\frac{k}{2}\lim_{h\to 0}\frac{\sinh}{h}=\frac{k}{2}.1=\frac{k}{2}$$

$$\therefore \lim_{x\to\frac{\pi}{2}} f(x)=f\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \frac{k}{2}=3$$

$$\Rightarrow k=6$$

Therefore, the required value of k is 6.

Question 27:

Find the values of k so that the function f is continuous at the indicated point.

$$f(x)=\begin{cases} kx^2, if\ x\le 2 \\ 3,\ if\ x>2 \end{cases} \text{at } x=2$$

Solution 27:

The given function is $f(x)=\begin{cases} kx^2, if\ x\le 2 \\ 3,\ \ if\ x>2 \end{cases}$

The given function f is continuous at $x=2$, if f is defined at $x=2$ and if the value of f at $x=2$ equals the limit of f at $x=2$

It is evident that f is defined at $x=2$ and $f(2)=k(2)^2=4k$

$\lim\limits_{x\to 2^-} f(x)=\lim\limits_{x\to 2^+} f(x)=f(2)$

$\Rightarrow \lim\limits_{x\to 2^-}\left(kx^2\right)=\lim\limits_{x\to 2^+}(3)=4k$

$\Rightarrow k \text{ x } 2^2 = 3 = 4k$

$\Rightarrow 4k = 3 = 4k$

$\Rightarrow 4k = 3$

$\Rightarrow k=\dfrac{3}{4}$

Therefore, the required value of k is $\dfrac{3}{4}$.

Question 28:
Find the values of k so that the function f is continuous at the indicated point. $f(x)=\begin{cases} kx+1, if\ x\le \pi \\ \cos x, if\ x>\pi \end{cases}$ at $x=\pi$

Solution 28:

The given function is $f(x)=\begin{cases} kx+1, if\ x\le \pi \\ \cos x, if\ x>\pi \end{cases}$

The given function f is continuous at $x=\pi$ and, if f is defined at $x=\pi$ and if the value of f at $x=\pi$ equals the limit of f at $x=\pi$

It is evident that f is defined at $x=\pi$ and $f(\pi)=k\pi+1$

$$\lim_{x\to\pi^-} f(x) = \lim_{x\to\pi^+} f(x) = f(\pi)$$

$$\Rightarrow \lim_{x\to\pi^-} (kx+1) = \lim_{x\to\pi^+} \cos x = k\pi + 1$$

$$\Rightarrow k\pi + 1 = \cos\pi = k\pi + 1$$

$$\Rightarrow k\pi + 1 = -1 = k\pi + 1$$

$$\Rightarrow k\pi + 1 = -1 = k\pi + 1$$

$$\Rightarrow k = -\frac{2}{\pi}$$

Therefore, the required value of k *is* $-\frac{2}{\pi}$.

Question 29:

Find the values of k so that the function f is continuous at the indicated point.

$$f(x) = \begin{cases} kx+1, \text{ if } x \le 5 \\ 3x-5, \text{ if } x > 5 \end{cases} \text{ at } x = 5$$

Solution 29:

The given function of f is $f(x) = \begin{cases} kx+1, \text{ if } x \le 5 \\ 3x-5, \text{ if } x > 5 \end{cases}$

The given function f is continuous at $x = 5$, if f is defined at $x = 5$ and if the value of f at $x = 5$ equals the limit of f at $x = 5$

It is evident that f is defined at $x = 5$ and $f(5) = kx + 1 = 5k + 1$

$$\lim_{x\to 5^-} f(x) = \lim_{x\to 5^+} f(x) = f(5)$$

$$\Rightarrow \lim_{x\to 5^-} (kx+1) = \lim_{x\to 5^+} (3x-5) = 5k + 1$$

$$\Rightarrow 5k + 1 = 15 - 5 = 5k + 1$$

$$\Rightarrow 5k + 1 = 10$$

$$\Rightarrow 5k = 9$$

$$\Rightarrow k = \frac{9}{5}$$

Therefore, the required value of k *is* $\frac{9}{5}$.

Question 30:

Find the values of a and b such that the function defined by $f(x)=\begin{cases} 5, & if\ x\le 2 \\ ax+b, & if\ 2<x<10 \\ 21 & if\ x\ge 10 \end{cases}$ is a continuous function.

Solution 30:

The given function f is $f(x)=\begin{cases} 5, & if\ x\le 2 \\ ax+b, & if\ 2<x<10 \\ 21 & if\ x\ge 10 \end{cases}$

It is evident that the given function f is defined at all points of the real line.
If f is a continuous function, then f is continuous at all real numbers.
In particular, f is continuous at $x=2$ and $x=10$
Since f is continuous at $x=2$, we obtain

$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^+} f(x)=f(2)$

$\Rightarrow \lim_{x\to 2^-}(5)=\lim_{x\to 2^+}(ax+b)=5$

$\Rightarrow 5=2a+b=5$

$\Rightarrow 2a+b=5$ ....(1)

Since f is a continuous at $x=10$, we obtain

$\lim_{x\to 10^-} f(x)=\lim_{x\to 10^+} f(x)=f(10)$

$\Rightarrow \lim_{x\to 10^-}(ax+b)=\lim_{x\to 10^+}(21)=21$

$\Rightarrow 10a+b-21=21$

$\Rightarrow 10a+b=21$ .....(2)

On subtracting equation (1) from equation (2), we obtain

$8a=16$

$\Rightarrow a=2$

By putting a = 2 in equation (1), we obtain

$2\times 2 + b = 5$

$\Rightarrow 4+b=5$

$\Rightarrow b=1$

Therefore, the values of a and b for which f is a continuous function are 2 and 1 respectively.

Question 31:

Show that the function defined by $f(x)=\cos(x^2)$ is a continuous function.

Solution 31:

The given function is $f(x)=\cos(x^2)$

This function f is defined for every real number and f can be written as the composition of two functions as,

$f=g\ o\ h,$ where $g(x)=\cos x \text{ and } h(x)=x^2$

$\left[\because (goh)(x)=g(h(x))=g(x^2)=\cos(x^2)=f(x)\right]$

It has to be first proved that $g(x)=\cos x \text{ and } h(x)=x^2$ are continuous functions.

It is evident that g is defined for every real number.

Let c be a real number.

Then, $g(c)=\cos c$

Put $x=c+h$

If $x\to c,$ then $h\to 0$

$\lim\limits_{x\to c} g(x)=\lim\limits_{x\to c}\cos x$

$=\lim\limits_{h\to 0}\cos(c+h)$

$=\lim\limits_{h\to 0}[\cos c\cosh-\sin c\sinh]$

$=\lim\limits_{h\to 0}\cos c\cosh-\lim\limits_{h\to 0}cinc\sinh$

$=\cos c\cos 0-\sin c\sin 0$

$=\cos c$ x 1-$\sin c$ x 0

=cosc

$\therefore \lim\limits_{x\to c} g(x)=g(c)$

Therefore, $g(x)=\cos x$ is a continuous function.

$h(x)=x^2$

Clearly, h is defined for every real number.

Let k be a real number, then $h(k)=k^2$

$\lim\limits_{x\to k} h(x)=\lim\limits_{x\to k} x^2=k^2$

$\therefore \lim\limits_{x\to k} h(x)=h(k)$

Therefore, h is a continuous function.

It is known that for real valued functions g and h, such that $(g\ o\ h)$ is defined at c, it g is continuous at c and it f is continuous at $g(c)$, then $(f\ o\ h)$ is continuous at c.

Therefore, $f(x)=(g\ o\ h)(x)=\cos(x^3)$ is a continuous function.

Question 32:

Show that the function defined by $f(x)=|\cos x|$ is a continuous function.

Solution 32:

The given function is $f(x)=|\cos x|$

This function f is defined for every real number and f can be written as the composition of two functions as,

$f=g\,o\,h$, where $g(x)=|x|$ and $h(x)=\cos x$

$\left[\because (goh)(x)=g(h(x))=g(\cos x)=|\cos x|=f(x)\right]$

It has to be first proved that $g(x)=|x|$ and $h(x)=\cos x$ are continuous functions.

$g(x)=|x|$, can be written as

$$g(x)=\begin{cases}-x, if\ \ x<0\\ \ \ x\ \ if\ \ x\geq 0\end{cases}$$

Clearly, g is defined for all real numbers.

Let c be a real number.

Case I :

If $c<0$, then $g(c)=-c$ and $\lim\limits_{x\to c} g(x)=\lim\limits_{x\to c}(-x)=-c$

$\therefore \lim\limits_{x\to c} g(x)=g(c)$

Therefore, g is continuous at all points x, such that $x<0$

Case II :

If $c>0$, then $g(c)=c$ and $\lim\limits_{x\to c} g(x)=\lim\limits_{x\to c} x=c$

$\therefore \lim\limits_{x\to c} g(x)=g(c)$

Therefore, g is continuous at all points x, such that $x>0$

Case III :

If $c=0$, then $g(c)=g(0)=0$

$\lim\limits_{x\to 0^-} g(x)=\lim\limits_{x\to 0^-}(-x)=0$

$\lim\limits_{x\to 0^+} g(x) = \lim\limits_{x\to 0^+}(x) = 0$

$\therefore \lim\limits_{x\to c^-} g(x) = \lim\limits_{x\to c^+} g(x) = g(0)$

Therefore, g is continuous at $x=0$

From the above three observations, it can be concluded that g is continuous at all points.

$h(x) = \cos x$

It is evident that $h(x) = \cos x$ is defined for every real number.

Let c be a real number. Put $x = c + h$

If $x \to c$, the $h \to 0$

$h(c) = \cos c$

$\lim\limits_{x\to c} h(x) = \lim\limits_{x\to c} \cos x$

$= \lim\limits_{h\to 0} \cos(c+h)$

$= \lim\limits_{h\to 0} [\cos c \cosh - \sin c \sinh]$

$= \lim\limits_{h\to 0} \cos c \cosh - \lim\limits_{h\to 0} \sin \sinh$

$= \cos c \cos 0 - \sin c \sin 0$

$= \cos c$ x $1 - \sin c$ x 0

$= \cos c$

$\therefore \lim\limits_{x\to c} h(x) = h(c)$

Therefore, $h(x) = \cos x$ is a continuous function.

It is known that fir real valued functions g and h, such that $(g\, o\, h)$ is defined at c, if g is continuous at c and if f is continuous at $g(c)$,then $(f\, o\, g)$ is continuous at c.

Therefore, $f(x) = (goh)(x) = g(h(x)) = g(cox) = |\cos x|$ is a continuous function.

Question 33:

Examine that $\sin|x|$ is a continuous function.

Solution 33:

Let $f(x) = \sin|x|$

This function f is defined for every real number and f cane be written as the composition of two functions as,

$f = g\, o\, h$, where $g(x) = |x|$ and $h(x) = \sin x$

$\left[\because (goh)(x) = g(h(x)) = g(\sin x) = |\sin x| = f(x)\right]$

It has to be prove first that $g(x) = |x|$ and $h(x) = \sin x$ are continuous functions.

$g(x) = |x|$ can be written as

$$g(x)\begin{cases} -x, if \ \ x < 0 \\ \ x \ \ if \ \ x \geq 0 \end{cases}$$

Clearly, g is defined for all real numbers.

Let c be a real number.

Case I :

If $c < 0$ $g(c) = -c$ and $\lim\limits_{x \to c} g(x) = \lim\limits_{x \to c}(-x) = -c$

$\therefore \lim\limits_{x \to c} g(x) = g(c)$

Therefore, g is continuous at all points x, that $x < 0$

Case II :

If $c > 0$, then $g(c) = c$ and $\lim\limits_{x \to c} g(x) = \lim\limits_{x \to c} x = c$

$\therefore \lim\limits_{x \to c} g(x) = g(c)$

Therefore, g is continuous at all points x, such that $x > 0$

Case III :

If $c = 0$, then $g(c) = g(0) = 0$

$\lim\limits_{x \to 0^-} g(x) = \lim\limits_{x \to 0^-}(-x) = 0$

$\lim\limits_{x \to 0^+} g(x) = \lim\limits_{x \to 0^+}(x) = 0$

$\therefore \lim\limits_{x \to 0^-} g(x) = \lim\limits_{x \to 0^+}(x) = g(0)$

Therefore, g is continuous at $x = 0$

From the above three observations, it can be concluded that g is continuous at all points.

$h(x) = \sin x$

It is evident that $h(x) = \sin x$ is defined for every real number.

Let c be a real number. Put $x = c + k$

If $x \to c$, then $k \to 0$

$h(c) = \sin c$

$\lim\limits_{x \to c} h(x) = \lim\limits_{x \to c} \sin x$

$= \lim_{k \to o} \sin(c+k)$

$= \lim_{k \to o} [\sin c \cos k + \cos c \sin k]$

$= \lim_{k \to o} (\sin c \cos k) + \lim_{h \to o} (\cos c \sin k)$

$= \sin c \cos 0 + \cos c \sin 0$

$= \sin c + 0$

$= \sin c$

$\therefore \lim_{x \to c} h(x) = g(c)$

Therefore, h is a continuous function,

It is known that for real valued functions g and h, such that $(g o h)$ is defined at c, if g is continuous at c and if f is continuous at $g(c)$, then $(f o h)$ is continuous at c.

Therefore, $f(x) = (goh)(x) = g(h(x)) = g(\sin x) = |\sin x|$ is a continuous function.

Question 34:

Find all the points of discontinuity of f defined by $f(x) = |x| - |x+1|$.

Solution 34:

The given function is $f(x) = |x| - |x+1|$.

The two functions, g and h, are defined as

$g(x) = |x|$ and $h(x) = |x+1|$

Then, $f = g - h$

The continuous of g and h is examined first.

$g(x) = |x|$ can be written as

$$g(x) = \begin{cases} -x, if & x < 0 \\ x, if & x \geq 0 \end{cases}$$

Clearly, g is defined for all real numbers.

Let c be a real number.

Case I :

If $c < 0$, then $g(c) = g(0) = -c$ and $\lim_{x \to c} g(x) = \lim_{x \to c} (-x) = -c$

$\therefore \lim_{x \to c} g(x) = g(c)$

Therefore, g is continuous at all points x, such that $x < 0$

Case II :

If $c>0$, then $g(c)=c \ \lim\limits_{x\to c} g(x)=\lim\limits_{x\to c} x=c$

$\therefore \lim\limits_{x\to c} g(x)=g(c)$

Therefore, g is continuous at all points x, such that $x>0$

Case III :

If $c=0$, then $g(c)=g(0)=0$

$\lim\limits_{x\to 0^-} g(x)=\lim\limits_{x\to 0^-}(-x)=0$

$\lim\limits_{x\to 0^+} g(x)=\lim\limits_{x\to 0^+} g(x)=0$

$\therefore \lim\limits_{x\to 0^-} g(x)=\lim\limits_{x\to 0^+}(x)=g(0)$

Therefore, g is continuous at $x=0$

From the above three observations, it can be concluded that g is continuous at all points.

$h(x)=|x+1|$ can be written as

$$h(x)=\begin{cases}-(x+1), if, x<-1\\ \quad x+1, \quad if, x\geq -1\end{cases}$$

Clearly, h is defined for every real number.

Let c be a real number.

Case I :

If $c<-1$, then $h(c)=-(c+1)$ and $\lim\limits_{x\to c} h(x)=\lim\limits_{x\to c}\left[-(x+1)\right]=-(c+1)$

$\therefore \lim\limits_{x\to c} h(x)=h(c)$

Therefore, h is continuous at all points x, such that $x<-1$

Case II :

If $c>-1$, then $h(c)=c+1$ and $\lim\limits_{x\to c} h(x)=\lim\limits_{x\to c}(x+1)=(c+1)$

$\therefore \lim\limits_{x\to c} h(x)=h(c)$

Therefore, h is continuous at all points x, such that $x>-1$

Case III :

If $c=-1$, then $h(c)=h(-1)=-1+1=0$

$\lim\limits_{x\to 1^-} h(x)=\lim\limits_{x\to 1^-}\left[-(x+1)\right]=-(-1+1)=0$

$\lim\limits_{x\to 1^+} h(x)=\lim\limits_{x\to 1^+}(x+1)=(-1+1)=0$

$\therefore \lim\limits_{x\to 1^-} h=\lim\limits_{x\to 1^+} h(x)=h(-1)$

Therefore, h is continuous at $x=-1$

From the above three observations, it can be concluded that h is continuous at all points of the real line.

g and h are continuous functions. Therefore, $f = g - h$ is also a continuous function.

Therefore, f has no point of discontinuity.

Exercise 5.2

Question 1:

Differentiate the function with respect to x.

$\sin\left(x^2+5\right)$

Solution 1:

Let $f(x) = \sin\left(x^2+5\right)$, $u(x) = x^2+5$, and $v(t) = \sin t$

Then, $(vou)(x) = v(u(x)) = v\left(x^2+5\right) = \tan\left(x^2+5\right) = f(x)$

Thus, f is a composite of two functions.

Put $t = u(x) = x^2+5$

Then, we obtain

$$\frac{dv}{dt} = \frac{d}{dt}(\sin t) = \cos t = \cos\left(x^2+5\right)$$

$$\frac{dt}{dx} = \frac{d}{dx}\left(x^2+5\right) = \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}(5) = 2x + 0 = 2x$$

Therefore, by chain rule. $\frac{df}{dx} = \frac{dv}{dt}.\frac{dt}{dx} = \cos\left(x^2+5\right) \text{x } 2x = 2x\cos\left(x^2+5\right)$

Alternate method

$$\begin{aligned}\frac{d}{dx}\left[\sin\left(x^2+5\right)\right] &= \cos\left(x^2+5\right)\cdot\frac{d}{dx}\left(x^2+5\right)\\ &= \cos\left(x^2+5\right).\left[\frac{d}{dx}\left(x^2\right)+\frac{d}{dx}(5)\right]\\ &= \cos\left(x^2+5\right).[2x+0]\\ &= 2x\cos\left(x^2+5\right)\end{aligned}$$

Question 2:

Differentiate the functions with respect of x.

$\cos(\sin x)$

Solution 2:

Let $f(x)=\cos(\sin x), u(x)=\sin x,$ and $v(t)=\cos t$

Then, $(vou)(x)=v(u(x))=v(\sin x)=\cos(\sin x)=f(x)$

Thus, f is a composite function of two functions.

Put $t=u(x)=\sin x$

$$\therefore \frac{dv}{dt}=\frac{d}{dt}[\cos t]=-\sin t=-\sin(\sin x)$$

$$\frac{dt}{dx}=\frac{d}{dx}(\sin x)=\cos x$$

By chain rule, $\dfrac{df}{dx}, \dfrac{dv}{dt}.\dfrac{dt}{dx}=-\sin(\sin x).\cos x=-\cos x\sin(\sin x)$

Alternate method

$$\frac{d}{dx}\left[\cos(\sin x)\right]=-\sin(sinx).\frac{d}{dx}(\sin x)=-\sin(\sin x)-\cos x=-\cos x\sin(\sin x)$$

Question 3:

Differentiate the functions with respect of x .

$\sin(ax+b)$

Solution 3:

Let $f(x)=\sin(ax+b), u(x)=ax+b,$ and $v(t)=\sin t$

Then, $(vou)(x)=v(u(x))=v(ax+b)=\sin(ax+b)=f(x)$

Thus, f is a composite function of two functions u and v.

Put $t=u(x)=ax+b$

Therefore,

$$\frac{dv}{dt}=\frac{d}{dt}(\sin t)=\cos t=\cos(ax+b)$$

$$\frac{dt}{dx}=\frac{d}{dx}(ax+b)=\frac{d}{dx}(ax)+\frac{d}{dx}(b)=a+0=a$$

Hence, by chain rule, we obtain

$$\frac{df}{dx}=\frac{dv}{dt}.\frac{dt}{dx}=\cos(ax+b)\cdot a=a\cos(ax+b)$$

Alternate method

$$\frac{d}{dx}\left[\sin(ax+b)\right]=\cos(ax+b).\frac{d}{dx}(ax+b)$$

$$=\cos(ax+b).\left[\frac{d}{dx}(ax)+\frac{d}{dx}(b)\right]$$

$$=\cos(ax+b).(a+0)$$

$$=a\cos(ax+b)$$

Question 4:

Differentiate the functions with respect of x.

$$\sec\left(\tan\left(\sqrt{x}\right)\right)$$

Solution 4:

Let $f(x)=\sec\left(\tan\left(\sqrt{x}\right)\right), u(x)=\sqrt{x},\ v(t)=\tan t$, and $w(s)=\sec s$

Then, $(wovou)(x)=w\left[v(u(x))\right]=w\left[v\left(\sqrt{x}\right)\right]=w\left(\tan\sqrt{x}\right)=\sec\left(\tan\sqrt{x}\right)=f(x)$

Thus, f is a composite function of three functions, u, v and w.

Put $s=v(t)=\tan t$ and $t=u(x)=\sqrt{x}$

Then, $\frac{dw}{ds}=\frac{d}{ds}(\sec s)=\sec s\tan s=\sec(\tan t).\tan(\tan t)\quad [s=\tan t]$

$$=\sec(\tan\sqrt{x}).\tan\left(\tan\sqrt{x}\right)\qquad [t=\sqrt{x}]$$

$$\frac{ds}{dt}=\frac{d}{dt}(\tan t)=\sec^2 t=\sec^2\sqrt{x}$$

$$\frac{dt}{dx}=\frac{d}{dx}\left(\sqrt{x}\right)=\frac{d}{dx}\left(x^{\frac{1}{2}}\right)=\frac{1}{2}.x^{\frac{1}{2}-1}=\frac{1}{2\sqrt{x}}$$

Hence, by chain rule, we obtain

$$\frac{dt}{dx}=\frac{dw}{ds}.\frac{ds}{dt}.\frac{dt}{dx}$$

$$=\sec\left(\tan\sqrt{x}\right)\bullet\tan\left(\tan\sqrt{x}\right)\text{ x }\sec^2\sqrt{x}\text{ x }\frac{1}{2\sqrt{x}}$$

$$=\frac{1}{2\sqrt{x}}\sec^2\sqrt{x}\left(\tan\sqrt{x}\right)\tan\left(\tan\sqrt{x}\right)$$

$$=\frac{\sec^2\sqrt{x}\sec\left(\tan\sqrt{x}\right)\tan\left(\tan\sqrt{x}\right)}{2\sqrt{x}}$$

Alternate method

$$\frac{d}{dx}\left[\sec\left(\tan\sqrt{x}\right)\right]=\sec\left(\tan\sqrt{x}\right).\tan\left(\tan\sqrt{x}\right).\frac{d}{dx}\left(\tan\sqrt{x}\right)$$

$$=\sec\left(\tan\sqrt{x}\right).\tan\left(\tan\sqrt{x}\right).\sec^2\left(\sqrt{x}\right).\frac{d}{dx}\left(\sqrt{x}\right).$$

$$=\sec\left(\tan\sqrt{x}\right).\tan\left(\tan\sqrt{x}\right).\sec^2\left(\sqrt{x}\right).\frac{1}{2\sqrt{x}}$$

$$=\frac{\sec\left(\tan\sqrt{x}\right).\tan\left(\tan\sqrt{x}\right).\sec^2\left(\sqrt{x}\right)}{2\sqrt{x}}$$

Question 5:

Differentiate the functions with respect of X.

$$\frac{\sin\left(ax+b\right)}{\cos\left(cx+d\right)}$$

Solution 5:

The given function is $f(x)=\frac{\sin(ax+b)}{\cos(cx+d)}=\frac{g(x)}{h(x)}$, where $g\left(x\right)=\sin\left(ax+b\right)$ and

$h\left(x\right)=\cos\left(cx+d\right)$

$$\therefore f^{'}=\frac{g^{'}h-gh^{'}}{h^2}$$

Consider $g\left(x\right)=\sin\left(ax+b\right)$

Let $u\left(x\right)=ax+b$, $v\left(t\right)=\sin t$

Then $\left(vou\right)\left(x\right)=v\left(u\left(x\right)\right)=v\left(ax+b\right)=\sin\left(ax+b\right)=g\left(x\right)$

$\therefore g$ is a composite function of two functions, u and v.

Put $t=u\left(x\right)=ax+b$

$$\frac{dv}{dt}=\frac{d}{dt}(\sin t)=\cos t=\cos(ax+b)$$

$$\frac{dt}{dx}=\frac{d}{dx}(ax+b)=\frac{d}{dx}(ax)+\frac{d}{dx}(b)=a+0=a$$

Therefore, by chain rule, we obtain

$$g^{'}=\frac{dg}{dx}=\frac{dv}{dt}.\frac{dt}{dx}=\cos(ax+b).a=a\cos(ax+b)$$

Consider $h(x)=\cos(cx+d)$

Let $p(x)=cx+d$, $q(y)=\cos y$

Then, $(qop)(x)=q(p(x))=q(cx+d)=\cos(cx+d)=h(x)$

$\therefore h$ is a composite function of two functions, p and q.

Put $y=p(x)=cx+d$

$$\frac{dq}{dy}=\frac{d}{dy}(\cos y)=-\sin y=-\sin(cx+d)$$

$$\frac{dy}{dx}=\frac{d}{dx}(cx+d)=\frac{d}{dx}(cx)+\frac{d}{dx}(d)=c$$

Therefore, by chain rule, we obtain

$$h^{'}=\frac{dh}{dx}=\frac{dq}{dy}.\frac{dy}{dx}=-\sin(cx+d)\text{xc}=-c\sin(cx+d)$$

$$\therefore f^{'}=\frac{a\cos(ax+b).\cos(cx+d)-\sin(ax+b)\{-c\sin cx+d\}}{[\cos(cx+d)]^2}$$

$$=\frac{a\cos(ax+b)}{\cos(cx+d)}+c\sin(ax+b)\cdot\frac{\sin(cx+d)}{\cos(cx+d)}\text{x}\frac{1}{\cos(cx+d)}$$

$$=a\cos(ax+b)\sec(cx+d)+c\sin(ax+b)\tan(cx+d)\sec(cx+d)$$

Question 6:

Differentiate the function with respect to x.

$\cos x^3\cdot\sin^2(x^5)$

Solution 6:

The given function is

$$\cos x^3 \cdot \sin^2\left(x^5\right)$$

$$\frac{d}{dx}\left[\cos x^3 . \sin^2\left(x^5\right)\right] = \sin^2\left(x^5\right) \text{x} \frac{d}{dx}\left(\cos x^3\right) + \cos x^3 \text{x} \frac{d}{dx}\left[\sin^2\left(x^5\right)\right]$$

$$= \sin^2\left(x^5\right) \text{x} \left(\text{-}\sin x^3\right) \text{x} \frac{d}{dx}\left(x^3\right) + \cos x^3 + 2\sin\left(x^5\right). \frac{d}{dx}\left[\sin x^5\right]$$

$$= \sin x^3 \sin^2\left(x^5\right) \text{x}\, 3x^2 + 2\sin x^5 \cos x^3 . \cos x^5 \;\; \text{x} \; \frac{d}{dx}\left(x^5\right)$$

$$= 3x^2 \sin x^3 . \sin^3\left(x^5\right) + 2\sin x^5 \cos x^5 \cos x^3 \;\; . \; \text{x}\, 5x^4$$

$$= 10x^4 \sin x^5 \cos x^5 \cos x^3 - 3x^2 \sin x^3 \sin^2\left(x^5\right)$$

Question 7:

Differentiate the functions with respect to x.

$$2\sqrt{\cot\left(x^2\right)}$$

Solution 7:

$$\frac{d}{dx}\left[2\sqrt{\cot\left(x^2\right)}\right]$$

$$= 2.\frac{1}{2\sqrt{\cot\left(x^2\right)}} \text{x} \frac{\text{d}}{\text{dx}}\left[\cot\left(x^2\right)\right]$$

$$= \sqrt{\frac{\sin\left(x^2\right)}{\cos\left(x^2\right)}} \text{x-cosec}^2\left(x^2\right) \text{x} \frac{d}{dx}\left(x^2\right)$$

$$= \sqrt{\frac{\sin\left(x^2\right)}{\cos\left(x^2\right)}} \text{x} \frac{1}{\sin^2\left(x^2\right)} \text{x} \left(2x\right)$$

$$= \frac{-2x}{\sqrt{\cos x^2 \sqrt{\sin x^2 \sin x^2}}}$$

$$= \frac{-2\sqrt{2}x}{\sqrt{2\sin x^2 \cos x^2} \sin x^2}$$

$$= \frac{-2\sqrt{2}x}{\sin x^2 \sqrt{\sin 2x^2}}$$

Question 8:
Differentiate the functions with respect to x
$\cos\left(\sqrt{x}\right)$

Solution 8:
Let $f(x)=\cos\left(\sqrt{x}\right)$
Also, let $u(x)=\sqrt{x}$
And, $v(t)=\cos t$
Then, $(vou)(x)=v(u(x))$
$=v\left(\sqrt{x}\right)$
$=\cos\sqrt{x}$
$=f(x)$
Clearly, f is a composite function of two functions, u and v, such that
$t=u(x)=\sqrt{x}$
Then,
$$\frac{dt}{dx}=\frac{d}{dx}\left(\sqrt{x}\right)=\frac{d}{dx}\left(x^{\frac{1}{2}}\right)$$
$$\frac{1}{2}x^{-\frac{1}{2}}=\frac{1}{2\sqrt{x}}$$
And, $\frac{dv}{dt}=\frac{d}{dt}(\cos t)=-\sin t \quad =-\sin\sqrt{x}$

By using chain rule, we obtain
$\frac{dt}{dx}=\frac{dv}{dt}\cdot\frac{dt}{dx}$
$=-\sin\left(\sqrt{x}\right)\cdot\frac{1}{2\sqrt{x}}$
$=-\frac{1}{2\sqrt{x}}\sin\left(\sqrt{x}\right)$
$=-\frac{\sin\left(\sqrt{x}\right)}{2\sqrt{x}}$

Alternate method

$$\frac{d}{dx}\left[\cos\left(\sqrt{x}\right)\right]=-\sin\left(\sqrt{x}\right).\frac{d}{dx}\left(\sqrt{x}\right)$$

$$=-\sin\left(\sqrt{x}\right) \text{ x } \frac{d}{dx}\left(x^{\frac{1}{2}}\right)$$

$$=-\sin\sqrt{x} \text{ x } \frac{1}{2}x^{-\frac{1}{2}}$$

$$=\frac{-\sin\sqrt{x}}{2\sqrt{x}}$$

Question 9:

Prove that the function f given by

$f(x)=|x-1|,\ x\in\mathbf{R}$ is not differentiable at $x=1$.

Solution 9:

The given function is $f(x)=|x-1|, x\in\mathbf{R}$

It is known that a function f is differentiable at a point $x=c$ in its domain if both

$\lim\limits_{k\to0^-}\frac{f(c+h)-f(c)}{h}$ and $\lim\limits_{h\to0^+}\frac{f(c+h)-f(c)}{h}$ are finite and equal.

To check the differentiability of the given function at $x=1$,

Consider the left hand limit of f at $x=1$

$$\lim_{h\to0^-}\frac{f(1+h)-f(1)}{h}=\lim_{h\to0^-}\frac{f|I+h-1||1-1|}{h}$$

$$=\lim_{h\to0^-}\frac{|h|-0}{h}=\lim_{h\to0^-}\frac{-h}{h} \qquad (h<0\Rightarrow|h|=-h)$$

$$=-1$$

Consider the right hand limit of f at $x=1$

$$\lim_{h\to0^+}\frac{f(1+h)-f(1)}{h}=\lim_{h\to0^+}\frac{f|I+h-1|-|1-1|}{h}$$

$$=\lim_{h\to0^+}\frac{|h|-0}{h}=\lim_{h\to0^+}\frac{h}{h} \qquad (h>0\Rightarrow|h|=h)$$

$$=1$$

Since the left and right hand limits of f at $x=1$ are not equal, f is not differentiable at $x=1$

Question 10:

Prove that the greatest integer function defined by $f=(x)=[x], 0<x<3$ is not differentiable at $x=1$ and $x=2$.

Solution 10:

The given function f is $f=(x)=[x], 0<x<3$

It is known that a function f is differentiable at a point $x=c$ in its domain if both $\lim_{h\to 0^-}\frac{f(c+h)-f(c)}{h}$ and $\lim_{h\to 0^+}\frac{f(c+h)-f(c)}{h}$ are finite and equal.

To check the differentiable of the given function at $x=1$, consider the left hand limit of f at $x=1$

$$\lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^-}\frac{[1+h]-[1]}{h}$$

$$=\lim_{h\to 0^-}\frac{0-1}{h}=\lim_{h\to 0^-}=\frac{-1}{h}=\infty$$

Consider the right hand limit of f at $x=1$

$$\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^+}\frac{[1+h][1]}{h}$$

$$=\lim_{h\to 0^+}\frac{1-1}{h}=\lim_{h\to 0^+}0=0$$

Since the left and right limits of f at $x=1$ are not equal, f is not differentiable at $x=1$

To check the differentiable of the given function at $x=2$, consider the left hand limit of f at $x=2$

$$\lim_{h\to 0^-}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0^-}\frac{[2+h]-[2]}{h}$$

$$=\lim_{h\to 0^-}\frac{1-2}{h}=\lim_{h\to 0^-}\frac{-1}{h}=\infty$$

Consider the right hand limit of f $at\, x=1$

$$\lim_{h\to 0^+}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0^+}\frac{[2+h]-[2]}{h}$$

$$=\lim_{h\to 0^+}\frac{1-2}{h}=\lim_{h\to 0^+}0=0$$

Since the left and right hand limits of f at $x=2$ are not equal, f is not differentiable at $x=2$

Exercise 5.3

Question 1:

Find $\frac{dy}{dx}: 2x+3y=\sin x$

Solution 1:

The given relationship is $2x+3y=\sin x$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dy}(2x+3y)=\frac{d}{dx}(\sin x)$$

$$\Rightarrow \frac{d}{dx}(2x)+\frac{d}{dx}(3y)=\cos x$$

$$\Rightarrow 2+3\frac{dy}{dx}=\cos x$$

$$\Rightarrow 3\frac{dy}{dx}=\cos x-2$$

$$\therefore \frac{dx}{dy}=\frac{\cos x-2}{3}$$

Question 2:

Find $\frac{dy}{dx}: 2x+3y=\sin y$

Solution 2:

The given relationship is $2x+3y=\sin y$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(2x)+\frac{d}{dx}(3y)=\frac{d}{dx}(\sin y)$$

$$\Rightarrow 2+3\frac{dy}{dx}=\cos y\frac{dy}{dx} \qquad [\text{By using chain rule}]$$

$$\Rightarrow 2=(\text{cosy-3})\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx}=\frac{2}{\cos y-3}$$

Question 3:

Find $\frac{dy}{dx}: ax+by^2=\cos y$

Solution 3:

The given relationship is $ax+by^2=\cos y$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(ax)+\frac{d}{dx}(by^2)=\frac{d}{dx}(\cos y)$$

$$\Rightarrow a+b\frac{d}{dx}(y^2)=\frac{d}{dx}(\cos y) \quad \ldots(1)$$

Using chain rule, we obtain $\frac{d}{dx}(y^2)=2y\frac{dy}{dx}$ and $\frac{d}{dx}(\cos y)=\sin y\frac{dy}{dx}$ ……….(2)

From (1) and (2), we obtain

$$a+bx\ 2y\frac{dy}{dx}=-\sin y\frac{dy}{dx}$$

$$\Rightarrow (2by+\sin y)\frac{dy}{dx}=-a$$

$$\therefore \frac{dy}{dx}=\frac{-a}{2by+\sin y}$$

Question 4:

Find $\frac{dy}{dx}: xy+y^2=\tan x+y$

Solution 4:

The given relationship is $xy+y^2=\tan x+y$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(xy+y^2)=\frac{d}{dx}(\tan x+y)$$

$$\Rightarrow \frac{d}{dx}(xy)+\frac{d}{dx}(y^2)=\frac{d}{dx}(\tan x)+\frac{dy}{dx}$$

$$\Rightarrow \left[y.\frac{d}{dx}(x)+x.\frac{dy}{dx}\right]+2y\frac{dy}{dx}=\sec^2 x+\frac{dy}{dx} \qquad [\text{using product rule and chain rule}]$$

$$\Rightarrow y.1 + x\frac{dy}{dx} + 2y\frac{dy}{dx} = \sec^2 x + \frac{dy}{dx}$$

$$\Rightarrow (x + 2y - 1)\frac{dy}{dx} = \sec^2 x - y$$

$$\therefore \frac{dy}{dx} = \frac{\sec^2 x - y}{(x + 2y - 1)}$$

Question 5:

Find $\frac{dy}{dx}: x^2 + xy + y^2 = 100$

Solution 5:

The given relationship is $x^2 + xy + y^2 = 100$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(x^2 + xy + y^2) = \frac{d}{dx}(100)$$

[Derivative of constant function is 0]

$$\Rightarrow \frac{d}{dx}(x^2) + \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = 0$$

$$\Rightarrow 2x + \left[y.\frac{d}{dx}(x) + x.\frac{dy}{dx}\right] + 2y\frac{dy}{dx} = 0$$ [Using product rule and chain rule]

$$\Rightarrow 2x + y.1 + x.\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow 2x + y + (x + 2y)\frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{2x + y}{x + 2y}$$

Question 6:

Find $\frac{dy}{dx}: x^2 + x^2 y + xy^2 + y^3 = 81$

Solution 6:

The given relationship is $x^2 + x^2 y + xy^2 + y^3 = 81$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(x^3 + x^2 y + xy^2 y^3) = \frac{d}{dx}(81)$$

$$\Rightarrow \frac{d}{dx}\left(x^3\right)+\frac{d}{dx}\left(x^2y\right)+\frac{d}{dx}\left(xy\right)^2+\frac{d}{dx}\left(y^3\right)=0$$

$$\Rightarrow 3x^2+\left[y\frac{d}{dx}\left(x^2\right)+x^2\frac{dy}{dx}\right]+\left[y^2\frac{d}{dx}\left(x\right)+x\frac{d}{dx}\left(y^2\right)\right]+3y^2\frac{dy}{dx}=0$$

$$\Rightarrow 3x^2+\left[y.2x+x^2\frac{dx}{dy}\right]+\left[y^2.1+x.2y.\frac{dy}{dx}\right]+3y^2\frac{dx}{dy}=0$$

$$\Rightarrow \left(x^2+2xy+3y^2\right)\frac{dy}{dx}+\left(3x^2+2xy+y^2\right)=0$$

$$\therefore \frac{dy}{dx}=\frac{-\left(3x^2+2xy+y^2\right)}{\left(x^2+2xy+3y^2\right)}$$

Question 7:

Find $\frac{dx}{dy}:\sin^2 y+\cos xy=\pi$

Solution 7:

The given relationship is $\sin^2 y+\cos xy=\pi$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}\left(\sin^2 y+\cos xy\right)=\frac{d}{dx}(\pi)$$

$$\Rightarrow \frac{d}{dx}\left(\sin^2 y\right)+\frac{d}{dx}\left(\cos xy\right)=0 \quad \text{.....(1)}$$

Using chain rule, we obtain

$$\frac{d}{dx}\left(\sin^2 y\right)=2\sin y\frac{d}{dx}\left(\sin y\right)=2\sin y\cos y\frac{dy}{dx} \quad \text{....(2)}$$

$$\frac{d}{dx}\left(\cos xy\right)=-\sin xy\frac{d}{dx}\left(xy\right)=-\sin xy\left[y\frac{d}{dx}\left(x\right)+x\frac{dy}{dx}\right]$$

$$=-\sin xy\left[y.1+x\frac{dy}{dx}\right]=-y\sin xy-x\sin xy\frac{dy}{dx} \quad \text{.....(3)}$$

From (1) ,(2) and (3), we obtain

$$2\sin y\cos y\frac{dy}{dx}-y\sin xy-x\sin xy\frac{dy}{dx}=0$$

$$\Rightarrow \left(2\sin y\cos y-x\sin xy\right)\frac{dy}{dx}=y\sin xy$$

$$\Rightarrow \left(\sin 2y-x\sin xy\right)\frac{dx}{dy}=y\sin xy$$

$$\therefore \frac{dx}{dy}=\frac{y\sin xy}{\sin 2y-x\sin xy}$$

Question 8:

Find $\frac{dy}{dx}: \sin^2 x+\cos^2 y=1$

Solution 8:

The given relationship is $\sin^2 x+\cos^2 y=1$

Differentiating this relationship with respect to x, we obtain

$$\frac{dy}{dx}\left(\sin^2 x+\cos^2 y\right)=\frac{d}{dx}(1)$$

$$\Rightarrow \frac{d}{dx}\left(\sin^2 x\right)+\frac{d}{dx}\left(\cos^2 y\right)=0$$

$$\Rightarrow 2\sin x.\frac{d}{dx}(\sin x)+2\cos y.\frac{d}{dx}(\cos y)=0$$

$$\Rightarrow 2\sin x\cos x+2\cos y(-\sin y).\frac{dy}{dx}=0$$

$$\Rightarrow \sin 2x-\sin 2y\frac{dy}{dx}=0$$

$$\therefore \frac{dx}{dy}=\frac{\sin 2x}{\sin 2y}$$

Question 9:

Find $\frac{dy}{dx}: y=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Solution 9:

The given relationship is $y=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$$y=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow \sin y=\frac{2x}{1+x^2}$$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(\sin y)=\frac{d}{dx}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow \cos y\frac{dy}{dx}=\frac{d}{dx}\left(\frac{2x}{1+x^2}\right) \qquad \ldots\ldots\ldots(1)$$

The function $\frac{2x}{1+x^2}$, is of the form of $\frac{u}{v}$.

Therefore, by quotient rule, we obtain

$$\frac{d}{dx}\left(\frac{2x}{1+x^2}\right)=\frac{\left(1+x^2\right)\frac{d}{dx}(2x)-2x.\frac{d}{dx}\left(1+x^2\right)}{\left(1+x^2\right)} \qquad \ldots\ldots.(2)$$

$$=\frac{\left(1+x^2\right).2-2x[0+2x]}{\left(1+x^2\right)^2}=\frac{2+2x^2-4x^3}{\left(1+x^2\right)^2}=\frac{2\left(1+x^2\right)}{\left(1+x^2\right)^2}$$

Also, $\sin y=\frac{2x}{1+x^2}$

$$\Rightarrow \cos y=\sqrt{1-\sin^2 y}=\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}=\sqrt{\frac{\left(1+x^2\right)^2-4x^2}{\left(1+x^2\right)^2}} \qquad \ldots\ldots\ldots\ldots(3)$$

$$=\sqrt{\frac{\left(1-x^2\right)^2}{\left(1-x^2\right)^2}}=\frac{1-x^2}{1+x^2}$$

From (1)(2) and (3), we obtain

$$\frac{1-x^2}{1+x^2}\text{ x }\frac{dy}{dx}=\frac{2\left(1-x^2\right)}{\left(1+x^2\right)^2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{2}{1+x^2}$$

Question 10:

Find $\frac{dx}{dy}: y=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right),-\frac{1}{\sqrt{3}}<x<\frac{1}{\sqrt{3}}$

Solution 10:

The given relationship is $y=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$

$$y=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$$

……..(1)

$$\Rightarrow \tan y=\frac{3x-x^3}{1-3x^2}$$

It is known that, $\tan y=\dfrac{3\tan\frac{y}{3}-\tan^3\frac{y}{3}}{1-3\tan^2\frac{y}{3}}$ ……(2)

Comparing equations (1) and (2), we obtain

$$x=\tan\frac{y}{3}$$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(x)=\frac{d}{dx}\left(\tan\frac{y}{3}\right)$$

$$\Rightarrow 1=\sec^2\frac{y}{3}.\frac{d}{dx}\left(\frac{y}{3}\right)$$

$$\Rightarrow 1=\sec^2\frac{y}{3}.\frac{1}{3}.\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}=\frac{3}{\sec^2\frac{y}{3}}=\frac{3}{1+\tan^2\frac{y}{3}}$$

$$\therefore \frac{dx}{dy}=\frac{3}{1+x^2}$$

Question 11:

Find $\dfrac{dy}{dx}:y\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right),0<x<1$

Solution 11:

The given relationship is,

$$y=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$\Rightarrow \cos y = \frac{1-x^2}{1+x^2}$

$\Rightarrow \frac{1-\tan^2 \frac{y}{2}}{1+\tan^2 \frac{y}{2}} = \frac{1-x^2}{1+x^2}$

On comparing L.H.S. and R.H.S. of the above relationship, we obtain

$\tan \frac{y}{2} = x$

Differentiating this relationship with respect to x, we obtain

$\sec^2 \frac{y}{2} \cdot \frac{d}{dx}\left(\frac{y}{2}\right) = \frac{d}{dx}(x)$

$\Rightarrow \sec^2 \frac{y}{2} \text{ x } \frac{1}{2} \frac{d}{dx} = 1$

$\Rightarrow \frac{dy}{dx} = \frac{2}{\sec^2 \frac{y}{2}}$

$\Rightarrow \frac{dy}{dx} = \frac{2}{1+\tan^2 \frac{y}{2}}$

$\therefore \frac{dy}{dx} = \frac{1}{1+x^2}$

Question 12:

Find $\frac{dy}{dx} : y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right), 0 < x < 1$

Solution 12:

The given relationship is $y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$\Rightarrow \sin y = \frac{1-x^2}{1+x^2}$

Differentiating this relationship with respect to x, we obtain

$\frac{d}{dx}(\sin y) = \frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right) \qquad \ldots\ldots(1)$

Using chain rule, we obtain

$$\frac{d}{dx}(\sin y) = \cos y.\frac{dy}{dx}$$

$$\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}$$

$$= \sqrt{\frac{\left(1+x^2\right)^2 - \left(1-x^2\right)^2}{\left(1+x^2\right)^2}} = \sqrt{\frac{4x^2}{\left(1+x^2\right)^2}} = \frac{2x}{1+x^2}$$

$$\therefore \frac{d}{dx}(\sin y) = \frac{2x}{1+x^2}\frac{dy}{dx} \quad \text{.......(2)}$$

$$\frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right) = \frac{\left(1+x^2\right)\left(1-x^2\right)' - \left(1-x^2\right)\left(1+x^2\right)'}{\left(1+x^2\right)^2} \quad [\text{using quotient rule}]$$

$$= \frac{\left(1+x^2\right)(-2x) - \left(1-x^2\right)(2x)}{\left(1+x^2\right)^2}$$

$$= \frac{-2x-2x^3-2x+2x^3}{\left(1+x^2\right)^2}$$

$$= \frac{-4x}{\left(1+x^2\right)^2} \quad \text{.....(3)}$$

From (1),(2), and (3), we obtain

$$\frac{2x}{1+x^2}\frac{dy}{dx} = \frac{-4x}{\left(1+x^2\right)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{1+x^2}$$

Alternate method

$$y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow \sin y = \frac{1-x^2}{1+x^2}$$

$$\Rightarrow \left(1+x^2\right)\sin y = 1-x^2$$

$$\Rightarrow \left(1+\sin y\right)x^2 = 1-\sin y$$

$$\Rightarrow x^2 = \frac{1-\sin y}{1+\sin y}$$

$$\Rightarrow x^2 = \frac{\left(\cos\frac{y}{2}-\sin\frac{y}{2}\right)^2}{\left(\cos\frac{y}{2}+\sin\frac{y}{x}\right)^2}$$

$$\Rightarrow x = \frac{\cos\frac{y}{2}-\sin\frac{y}{2}}{\cos\frac{y}{2}+\sin\frac{y}{2}}$$

$$\Rightarrow x = \frac{1-\tan\frac{y}{2}}{1+\tan\frac{y}{2}}$$

$$\Rightarrow x = \tan\left(\frac{\pi}{4}-\frac{\pi}{2}\right)$$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(x) = \frac{d}{dx}.\left[\tan\left(\frac{\pi}{4}-\frac{y}{2}\right)\right]$$

$$\Rightarrow 1 = \sec^2\left(\frac{\pi}{4}-\frac{y}{2}\right).\frac{d}{dx}\left(\frac{\pi}{4}-\frac{y}{2}\right)$$

$$\Rightarrow 1 = \left[1+\tan^2\left(\frac{\pi}{4}-\frac{y}{2}\right).\left(-\frac{1}{2}.\frac{dy}{dx}\right)\right.$$

$$\Rightarrow 1 = \left(1+x^2\right)\left(-\frac{1}{2}\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dx}{dy} = \frac{-2}{1+x^2}$$

Question 13:

Find $\frac{dy}{dx}: y=\cos^{-1}\left(\frac{2x}{1+x^2}\right), -1<x<1$

Solution 13:

The given relationship is $y=\cos^{-1}\left(\frac{2x}{1+x^2}\right)$

$$y=\cos^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow \cos y=\frac{2x}{1+x^2}$$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(\cos y)=\frac{d}{dx}.\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow -\sin y.\frac{dy}{dx}=\frac{\left(1+x^2\right).\frac{d}{dx}(2x)-2x.\frac{d}{dx}\left(1+x^2\right)}{\left(1+x^2\right)^2}$$

$$\Rightarrow -\sqrt{1-\cos^2 y}\,\frac{dy}{dx}=\frac{\left(1+x^2\right)\text{x 2-2x.2x}}{\left(1+\text{x}^2\right)^2}$$

$$\Rightarrow \left[\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}\right]\frac{dy}{dx}=-\left[\frac{2\left(1-x\right)^2}{\left(1+x^2\right)^2}\right]$$

$$\Rightarrow \sqrt{\frac{\left(1-x^2\right)^2-4x^2}{\left(1+x^2\right)^2}}\,\frac{dy}{dx}=\frac{-2(1-x^2)}{(1+x^2)}$$

$$\Rightarrow \sqrt{\frac{\left(1-x^2\right)^2}{\left(1+x^2\right)^2}}\,\frac{dy}{dx}=\frac{-2\left(1-x^2\right)}{\left(1-x^2\right)^2}$$

$$\Rightarrow \frac{1-x^2}{1+x^2}.\frac{dy}{dx}=\frac{-2\left(1-x^2\right)}{\left(1+x^2\right)^2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{-2}{1+x^2}$$

Question 14:

Find $\frac{dy}{dx}: y=\sin^{-1}\left(2x\sqrt{1-x^2}\right), -\frac{1}{\sqrt{2}}<x<\frac{1}{\sqrt{2}}$

Solution 14:

Relationship is $y=\sin^{-1}\left(2x\sqrt{1-x^2}\right)$

$$y=\sin^{-1}\left(2x\sqrt{1-x^2}\right)$$

$$\Rightarrow \sin y = 2x\sqrt{1-x^2}$$

Differentiating this relationship with respect to x, we obtain

$$\cos y = \frac{dy}{dx} = 2\left[x\frac{d}{dx}\left(\sqrt{1-x^2}\right)+\sqrt{1-x^2}\,\frac{dx}{dx}\right]$$

$$\Rightarrow \sqrt{1-\sin^2 y}\,\frac{dy}{dx} = 2\left[\frac{x}{2}.\frac{-2x}{\sqrt{1-x^2}}+\sqrt{1-x^2}\right]$$

$$\Rightarrow \sqrt{1-\left(2x\sqrt{1-x^2}\right)^2}\,\frac{dy}{dx} = 2\left[\frac{-x^2+1-x^2}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow \sqrt{1-4x^2\left(1-x^2\right)}\,\frac{dy}{dx} = 2\left[\frac{1-2x^2}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow \sqrt{\left(1-2x\right)^2}\,\frac{dy}{dx} = 2\left[\frac{1-2x^2}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow \left(1-2x^2\right)\frac{dy}{dx} = 2\left[\frac{1-2x^2}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}}$$

Question 15:

Find $\frac{dy}{dx}$: $y=\sec^{-1}\left(\frac{1}{2x^2-1}\right), 0<x<\frac{1}{\sqrt{2}}$

Solution 15:

The given relationship is $y=\sec^{-1}\left(\frac{1}{2x^2-1}\right)$

$$y=\sec^{-1}\left(\frac{1}{2x^2-1}\right)$$

$$\Rightarrow \sec y = \frac{1}{2x^2 - 1}$$

$$\Rightarrow \cos y = 2x^2 - 1$$

$$\Rightarrow 2x^2 = 1 + \cos y$$

$$\Rightarrow 2x^2 = 2\cos^2 \frac{y}{2}$$

$$\Rightarrow x = \cos \frac{y}{2}$$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(x) = \frac{d}{dx}\left(\cos \frac{y}{2}\right)$$

$$\Rightarrow 1 = \sin \frac{y}{2} . \frac{d}{dx}\left(\frac{y}{2}\right)$$

$$\Rightarrow \frac{-1}{\sin \frac{y}{2}} = \frac{1}{2}\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{\sin \frac{y}{2}} = \frac{-2}{\sqrt{1 - \cos^2 \frac{y}{2}}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{\sqrt{1 - x^2}}$$

Exercise 5.4

Question 1:

Differentiating the following $w.r.t.\ x$: $\frac{e^x}{\sin x}$

Solution 1:

Let $y = \frac{e^x}{\sin x}$

differentiating w.r.t x, we obtain

$$\frac{dy}{dx}=\frac{\sin x\frac{d}{dx}\left(e^x\right)-e^x\frac{d}{dx}\left(\sin x\right)}{\sin^2 x}$$
$$=\frac{\sin x.\left(e^x\right)-e^x.\left(\cos x\right)}{\sin^2 x}$$
$$=\frac{e^x\left(\sin x-\cos x\right)}{\sin^2 x},x\neq n\pi,n\in\mathbf{Z}$$

Question 2:

Differentiating the following $e^{\sin^{-1}x}$

Solution 2:

Let $y=e^{\sin^{-1}x}$

differentiating w.r.t x , we obtain

$$\frac{dy}{dx}=\frac{d}{dx}\left(e^{\sin^{-1}x}\right)$$
$$\Rightarrow\frac{dy}{dx}=e^{\sin^{-1}x}.\frac{d}{dx}\left(\sin^{-1}x\right)$$
$$\Rightarrow e^{\sin^{-1}x}.\frac{1}{\sqrt{1-x^2}}$$
$$\Rightarrow\frac{e\sin^{-1}x}{\sqrt{1-x^2}}$$
$$\therefore\frac{dy}{dx}=\frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}},x\in\left(-1,1\right)$$

Question 3:

Differentiating the following $w.r.t.\ x:\ e^{x^3}$

Solution 3:

Let $y=e^{x^3}$

By using the quotient rule, we obtain

$$\frac{dy}{dx}=\frac{d}{dx}=\left(e^{x^3}\right)=e^{x^3}.\frac{d}{dx}\left(x^3\right)=e^{x^3}.3x^2=3x^2e^{x^3}$$

Question 4:

Differentiating the following $w.r.t.\ x$: $\sin\left(\tan^{-1} e^{-x}\right)$

Solution 4:

Let $y = \sin\left(\tan^{-1} e^{-x}\right)$

By using the chain rule, we obtain

$$\frac{dy}{dx} : \frac{d}{dx}\left[\sin\left(\tan^{-1} e^{-x}\right)\right]$$

$$= \cos\left(\tan^{-1} e^{-x}\right).\frac{d}{dx}\left(\tan^{-1} e^{-x}\right)$$

$$= \cos\left(\tan^{-1} e^{-x}\right).\frac{1}{1+\left(e^{-x}\right)^2}.\frac{d}{dx}\left(e^{-x}\right)$$

$$= \frac{\cos\left(\tan^{-1} e^{-x}\right)}{1+e^{-2x}}.e^{-x}.\frac{d}{dx}(-x)$$

$$= \frac{e^{-x}\cos\left(\tan^{-1} e^{-x}\right)}{1+e^{-2x}}\text{x}(\text{-}1)$$

$$= \frac{-e^{-x}\cos\left(\tan^{-1} e^{-x}\right)}{1+e^{-2x}}$$

Question 5:

Differentiating the following $w.r.t.\ x$: $\log\left(\cos e^{x}\right)$

Solution 5:

Let $y = \log\left(\cos e^{x}\right)$

By using the chain rule, we obtain

$$\frac{dy}{dx}=\frac{d}{dx}\left[\log\left(\cos e^{x}\right)\right]$$
$$=\frac{1}{\cos e^{x}}\cdot\frac{d}{dx}\left(\cos e^{x}\right)$$
$$=\frac{1}{\cos e^{x}}\cdot\left(-\sin e^{x}\right)\cdot\frac{d}{dx}\left(e^{x}\right)$$
$$=\frac{-\sin e^{x}}{\cos e^{x}}\cdot e^{x}$$
$$=-e^{x}\tan e^{x},\ \ e^{x}\neq(2n+1)\frac{\pi}{2}, n\in\mathbf{N}$$

Question 6:

Differentiating the following $w.r.t.\ x$: $e^{x}+e^{x^{2}}+\ldots+e^{x^{5}}$

Solution 6:

$$\frac{d}{dx}\left(e^{x}+e^{x^{2}}+\ldots+e^{x^{5}}\right)$$
$$=\frac{d}{dx}\left(e^{x}\right)+\frac{d}{dx}\left(e^{x^{2}}\right)+\frac{d}{dx}\left(e^{x^{3}}\right)+\frac{d}{dx}\left(e^{x^{4}}\right)+\frac{d}{dx}\left(e^{x^{5}}\right)$$
$$=e^{x}+\left[e^{x^{2}}\text{x}\frac{d}{dx}\left(x^{2}\right)\right]+\left[e^{x^{3}}\text{x}\frac{d}{dx}\left(x^{3}\right)\right]+\left[e^{x^{4}}\text{x}\frac{d}{dx}\left(x^{4}\right)\right]+\left[e^{x^{5}}\text{x}\frac{d}{dx}\left(x^{5}\right)\right]$$
$$=e^{x}+\left(e^{x^{2}}\text{x}2x\right)+\left(e^{x^{3}}\text{x}3x^{2}\right)+\left(e^{x^{4}}\text{x}4x^{3}\right)+\left(e^{x^{5}}\text{x}5x^{4}\right)$$
$$=e^{x}+2xe^{x^{2}}+3x^{2}e^{x^{3}}+4x^{3}e^{x^{4}}+5x^{4}e^{x^{5}}$$

Question 7:

Differentiating the following $w.r.t.\ x$: $\sqrt{e^{\sqrt{x}}}, x>0$

Solution 7:

Let $y=\sqrt{e^{\sqrt{x}}}$

Then, $y^{2}=e^{\sqrt{x}}$

By Differentiating this relationship with respect to x , we obtain

$$y^{2}=e^{\sqrt{x}}$$
$$\Rightarrow 2y\frac{dy}{dx}=e^{\sqrt{x}}\frac{d}{dx}\left(\sqrt{x}\right) \qquad [\text{By applying the chain rule}]$$

$$\Rightarrow 2y\frac{dy}{dx}=e^{\sqrt{x}}\frac{1}{2}.\frac{1}{\sqrt{x}}$$
$$\Rightarrow \frac{dy}{dx}=\frac{e^{\sqrt{x}}}{4y\sqrt{x}}$$
$$\Rightarrow \frac{dy}{dx}=\frac{e^{\sqrt{x}}}{4\sqrt{e^{\sqrt{x}}}\sqrt{x}}$$
$$\Rightarrow \frac{dy}{dx}=\frac{e^{\sqrt{x}}}{4\sqrt{xe^{\sqrt{x}}}}, x>0$$

Question 8:

Differentiating the following $w.r.t.\ x: l\text{og}\left(l\text{og}\,x\right), x>1$

Solution 8:

Let $y=l\text{og}\left(l\text{og}\,x\right)$

By using the chain rule, we obtain

$$\frac{dy}{dx}=\frac{d}{dx}\left[l\text{og}\left(l\text{og}\,x\right)\right]$$
$$=\frac{1}{l\text{og}\,x}.\frac{d}{dx}\left(l\text{og}\,x\right)$$
$$=\frac{1}{l\text{og}\,x}.\frac{1}{x}$$
$$\frac{1}{x\log x}, x>1$$

Question 9:

Differentiating the following $w.r.t.\ x: \frac{\cos x}{\log x}, x>0$

Solution 9:

Let $y=\frac{\cos x}{\log x}$

By using the quotient rule, we obtain

$$\frac{dy}{dx}=\frac{\frac{d}{dx}(\cos x)\text{x }\log x\text{-}\cos x\text{ x}\frac{d}{dx}(\log x)}{(\log x)^2}$$

$$=\frac{-\sin x\log x-\cos x\text{ x}\frac{1}{x}}{(\log x)^2}$$

$$=\frac{-[x\log x.\sin x+\cos x]}{x(\log x)^2},x>0$$

Question 10:

Differentiating the following *w.r.t.* $x: \cos\left(\log x+e^x\right), x>0$

Solution 10:

Let $y=\cos\left(\log x+e^x\right)$

By using the chain rule, we obtain

$$y=\cos\left(\log x+e^x\right)$$

$$\frac{dy}{dx}=-\sin\left[\log x+e^x\right].\frac{d}{dx}\left(\log x+e^x\right)$$

$$=\sin\left(\log x+e^x\right).\left[\frac{d}{dx}(\log x)+\frac{d}{dx}\left(e^x\right)\right]$$

$$=-\sin\left(\log x+e^x\right).\left(\frac{1}{x}+e^x\right)$$

$$=\left(\frac{1}{x}+e^x\right)\sin\left(\log x+e^x\right),x>0$$

Exercise 5.5

Question 1:

Differentiate the following with respect to x.

$\cos x.\cos 2x.\cos 3x$

Solution 1:

Let $y=\cos x.\cos 2x.\cos 3x$

Taking logarithm or both the side, we obtain

$$\log y=\log\left(\cos x.\cos 2x.\cos 3x\right)$$

$$\Rightarrow \log y=\log\left(\cos x\right)+\log\left(\cos 2x\right)+\log(\cos 3x)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{\cos x}.\frac{d}{dx}\left(\cos x\right)+\frac{1}{\cos 2x}.\frac{d}{dx}\left(\cos 2x\right)+\frac{1}{\cos 3x}.\frac{d}{dx}\left(\cos 3x\right)$$

$$\Rightarrow \frac{dy}{dx}=y\left[-\frac{\sin x}{\cos x}-\frac{\sin 2x}{\cos 2x}.\frac{d}{dx}\left(2x\right)-\frac{\sin 3x}{\cos 3x}.\frac{d}{dx}\left(3x\right)\right]$$

$$\therefore \frac{dy}{dx}=-\cos x.\cos 2x.\cos 3x\left[\tan x+2\tan 2x+3\tan 3x\right]$$

Question 2:

Differentiate the function with respect to x.

$$\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$$

Solution 2:

Let $y=\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$

Taking logarithm or both the side, we obtain

$$\log y=\log\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$$

$$\Rightarrow \log y=\frac{1}{2}\log\left[\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}\right]$$

$$\Rightarrow \log y=\frac{1}{2}\left[\log\{(x-1)(x-2)\}-\log\{(x-3)(x-4)(x-5)\}\right]$$

$$\Rightarrow \log y=\frac{1}{2}\left[\log(x-1)+\log(x-2)-\log(x-3)-\log(x-4)-\log(x-5)\right]$$

Differentiating both sides with respect to , we obtain

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{2}\left[\begin{array}{c}\frac{1}{x-1}.\frac{d}{dx}(x-1)+\frac{1}{x-2}.\frac{d}{dx}(x-2)-\frac{1}{x-3}.\frac{d}{dx}(x-3)\\-\frac{1}{x-4}.\frac{d}{dx}(x-4)-\frac{1}{x-5}.\frac{d}{dx}(x-5)\end{array}\right]$$

$$\Rightarrow\frac{dy}{dx}=\frac{y}{2}\left(\frac{1}{x-1}+\frac{1}{x-2}-\frac{1}{x-3}-\frac{1}{x-4}-\frac{1}{x-5}\right)$$

$$\therefore\frac{dy}{dx}=\frac{1}{2}\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\left[\frac{1}{x-1}+\frac{1}{x-2}-\frac{1}{x-3}-\frac{1}{x-4}-\frac{1}{x-5}\right]$$

Question 3:
Differentiate the function with respect to x.

$(\log x)^{\cos x}$

Solution 3:
Let $y=(\log x)^{\cos x}$

Taking logarithm or both the side, we obtain

$\log y=\cos x.\log(\log x)$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}.\frac{dy}{dx}=\frac{d}{dx}(\cos x)\,\mathrm{x}\log(\log x)+\cos x\ \mathrm{x}\frac{d}{dx}[\log(\log x)]$$

$$\Rightarrow\frac{1}{y}.\frac{dy}{dx}=-\sin x\log(\log x)+\cos x\ \mathrm{x}\frac{1}{\mathrm{logx}}.\frac{d}{dx}(\log x)$$

$$\Rightarrow\frac{dy}{dx}=y\left[-\sin x\log(\log x)+\frac{\cos x}{\log x}\ \mathrm{x}\ \frac{1}{x}\right]$$

$$\therefore\frac{dy}{dx}=(\log x)^{\cos x}\left[\frac{\cos x}{x\log x}-\sin \mathrm{x}\log(\log x)\right]$$

Question 4:
Differentiate the function with respect to x.

$$x^x-2^{\sin x}$$

Solution 4:

Let $y = x^x - 2^{\sin x}$

Also, let $x^x = u$ and $2^{\sin x} = v$

$\therefore y = u - v$

$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx}$

$$u = x^x$$

Taking logarithm on both sides, we obtain

$\log u = x \log x$

Differentiating both sides with respect to x, we obtain

$\frac{1}{u}\frac{du}{dx} = \left[\frac{d}{dx}(x) \text{ x } \log x + x \text{ x } \frac{d}{dx}(\log x)\right]$

$\Rightarrow \frac{du}{dx} = u\left[1 \text{ x } \log x + x \text{ x } \frac{1}{x}\right]$

$\Rightarrow \frac{du}{dx} = x^x(\log x + 1)$

$\Rightarrow \frac{du}{dx} = x^x(1 + \log x)$

$v = 2^{\sin x}$

Taking logarithm on both the sides with respect to x, we obtain

$\log v = \sin x . \log 2$

Differentiating both sides with respect to x, we obtain

$\frac{1}{v} . \frac{dv}{dx} = \log 2 . \frac{d}{dx}(\sin x)$

$\Rightarrow \frac{dv}{dx} = v \log 2 \cos x$

$\Rightarrow \frac{dv}{dx} = 2^{\sin x} \cos x \log 2$

$\therefore \frac{dy}{dx} = x^2(1 + \log x) - 2^{\sin x} \cos x \log 2$

Question 5:

Differentiate the function with respect to x.

$(x+3)^2 \cdot (x+4)^3 \cdot (x+5)^4$

Solution 5:

Let

$$y=(x+3)^2\cdot(x+4)^3\cdot(x+5)^4$$

Taking logarithm on both sides, we obtain.

$$\log y=\log(x+3)^2+\log(x+4)^3+\log(x+5)^4$$

$$\Rightarrow \log y=2\log(x+3)+3\log(x+4)+4\log(x+5)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}.\frac{dy}{dx}=2.\frac{1}{x+3}.\frac{d}{dx}(x+3)+3.\frac{1}{x+4}.\frac{d}{dx}(x+4)+4.\frac{1}{x+5}.\frac{d}{dx}(x+5)$$

$$\Rightarrow \frac{dy}{dx}=y\left[\frac{2}{x+3}+\frac{3}{x+4}+\frac{4}{x+5}\right]$$

$$\Rightarrow \frac{dy}{dx}=(x+3)^2(x+4)^3(x+5)^4\cdot\left[\frac{2}{x+3}+\frac{3}{x+4}+\frac{4}{x+5}\right]$$

$$\Rightarrow \frac{dy}{dx}=(x+3)^2(x+4)^3(x+5)^4\cdot\left[\frac{2(x+4)(x+5)+3(x+3)(x+5)+4(x+3)(x+4)}{(x+3)(x+4)(x+5)}\right]$$

$$\Rightarrow \frac{dy}{dx}=(x+3)(x+4)^2(x+5)^3\cdot\left[2(x^2+9x+20)+3(x^2+9x+15)+4(x^2+7x+12)\right]$$

$$\therefore \frac{dy}{dx}=(x+3)(x+4)^2(x+5)^3(9x^2+70x+133)$$

Question 6:

Differentiate the function with respect to x.

$$\left(x+\frac{1}{x}\right)^x+x^{\left(1+\frac{1}{x}\right)}$$

Solution 6:

Let $y=\left(x+\frac{1}{x}\right)^x+x^{\left(1+\frac{1}{x}\right)}$

Also, let $u=\left(x+\frac{1}{x}\right)^x$ and $v=x^{\left(1+\frac{1}{x}\right)}$

$\therefore y=u+v$

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \qquad \text{.......(1)}$$

Then, $u=\left(x+\frac{1}{x}\right)^x$

Taking log on both sides

$$\Rightarrow \log u = \log\left(x+\frac{1}{x}\right)^x$$

$$\Rightarrow \log u = x\log\left(x+\frac{1}{x}\right)$$

Differentiating both sides with respect to x , we obtain

$$\frac{1}{u}\frac{du}{dx} = \frac{d}{dx}(x)\times\log\left(x+\frac{1}{x}\right)+x\times\frac{d}{dx}\left[\log\left(x+\frac{1}{x}\right)\right]$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = 1\text{x}\log\left(x+\frac{1}{x}\right)+x\text{x}\frac{1}{\left(x+\frac{1}{x}\right)}\cdot\frac{d}{dx}\left(x+\frac{1}{x}\right)$$

$$\Rightarrow \frac{du}{dx} = u\left[\log\left(x+\frac{1}{x}\right)+\frac{x}{\left(x+\frac{1}{x}\right)}x\left(x+\frac{1}{x^2}\right)\right]$$

$$\Rightarrow \frac{du}{dx} = \left(x+\frac{1}{x}\right)^x\left[\log\left(x+\frac{1}{x}\right)+\frac{\left(x-\frac{1}{x}\right)}{\left(x+\frac{1}{x}\right)}\right]$$

$$\Rightarrow \frac{du}{dx} = \left(x+\frac{1}{x}\right)^x\left[\log\left(x+\frac{1}{x}\right)+\frac{x^2-1}{x^2+1}\right]$$

$$\Rightarrow \frac{du}{dx} = \left(x+\frac{1}{x}\right)\left[\frac{x^2-1}{x^2+1}+\log\left(x+\frac{1}{x}\right)\right] \qquad \text{….(2)}$$

$$v = x^{\left(x+\frac{1}{x}\right)}$$

Taking log on both sides, we obtain

$$\log v = \log x^{\left(1+\frac{1}{x}\right)}$$

$$\Rightarrow \log v = \left(1+\frac{1}{x}\right)\log x$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}\cdot\frac{dv}{dx}=\left[\frac{d}{dx}\left(1+\frac{1}{x}\right)\right]\text{x}\ \log x+\left(1+\frac{1}{x}\right)\cdot\frac{d}{dx}\log x$$

$$\Rightarrow\frac{1}{v}\frac{dv}{dx}=\left(-\frac{1}{x^2}\right)\log x+\left(1+\frac{1}{x}\right)\cdot\frac{1}{x}$$

$$\Rightarrow\frac{1}{v}\frac{dv}{dx}=-\frac{\log x}{x^2}+\frac{1}{x}+\frac{1}{x^2}$$

$$\Rightarrow\frac{dv}{dx}=v\left[\frac{-\log x+x+1}{x^2}\right]$$

$$\Rightarrow\frac{dv}{dx}=x^{\left(1+\frac{1}{x}\right)}\left(\frac{x+1-\log x}{x^2}\right)\qquad\text{.........(3)}$$

Therefore, from (1),(2)and (3), we obtain

$$\frac{dy}{dx}=\left(x+\frac{1}{x}\right)^x\left[\frac{x^2-1}{x^2+}+\log\left(x+\frac{1}{x}\right)\right]+x^{\left(x+\frac{1}{x}\right)}\left(\frac{x+1-\log x}{x^2}\right)$$

Question 7:

Differentiate the function with respect to x.

$(\log x)^x+x^{\log x}$

Solution 7:

Let $y=(\log x)^x+x^{\log x}$

Also, let $u=(\log x)^x$ and $v=x^{\log x}$

$\therefore y=u+v$

$$\Rightarrow\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}\qquad\text{........(1)}$$

$u=(\log x)^x$

$\Rightarrow\log u=\log\left[(\log x)^x\right]$

$\Rightarrow\log u=x\log(\log x)$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx}=\frac{d}{dx}(x)x\log(\log x)+x.\frac{d}{dx}[\log(\log x)]$$

$$\Rightarrow \frac{du}{dx}=u\left[1x\log(\log x)+x.\frac{1}{\log x}.\frac{d}{dx}(\log x)\right]$$

$$\Rightarrow \frac{du}{dx}=(\log x)^{x}\left[\log(\log x)+\frac{x}{\log x}.\frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx}=(\log x)^{x}\left[\log(\log x)+\frac{1}{\log x}\right]$$

$$\Rightarrow \frac{du}{dx}=(\log x)^{x}\left[\frac{\log(\log x).\log x+1}{\log x}\right]$$

$$\frac{du}{dx}=(\log x)^{x-1}\left[1+\log x.\log(\log x)\right] \qquad \ldots\ldots(2)$$

$$v=x^{\log x}$$

$$\Rightarrow \log v=\log\left(x^{\log x}\right)$$

$$\Rightarrow \log v=\log x\log x=(\log x)^{2}$$

Differentiating both sides with respect to x , we obtain

$$\frac{1}{v}.\frac{dv}{dx}=\frac{d}{dx}\left[(\log x)^{2}\right]$$

$$\Rightarrow \frac{1}{v}.\frac{dv}{dx}=2(\log x).\frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{dv}{dx}=2x^{\log x}\frac{\log x}{x}$$

$$\Rightarrow \frac{dv}{dx}=2x^{\log x-1}.\log x \qquad \ldots.(3)$$

Therefore, from (1),(2), and (3), we obtain

$$\frac{dy}{dx}=(\log x)^{x-1}[1+\log x.\log(\log x)]+2x^{\log x-1}\cdot\log x$$

Question 8:

Differentiate the function with respect to x

$$(\sin x)^{x}+\sin^{-1}\sqrt{x}$$

Solution 8:

Let $y=(\sin x)^{x}+\sin^{-1}\sqrt{x}$

Also, let $u=(\sin x)^x$ and $v=\sin^{-1}\sqrt{x}$

$\therefore y=u+v$

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}-\frac{dv}{dx} \qquad \ldots\ldots..(1)$$

$u=(\sin x)^x$

$\Rightarrow \log u=\log(\sin x)^x$

$\Rightarrow \log u=x\log(\sin x)$

Differentiating both sides with respect to x, we obtain

$$\Rightarrow \frac{1}{u}\frac{du}{dx}=\frac{d}{dx}(x)\text{x }\log(\sin x)+x\text{ x }\frac{\mathrm{d}}{dx}[\log(\sin x)]$$

$$\Rightarrow \frac{du}{dx}=u\left[1.\log(\sin x)+x.\frac{1}{\sin x}.\frac{d}{dx}(\sin x)\right]$$

$$\Rightarrow \frac{du}{dx}=(\sin x)^x\left[\log(\sin x)+\frac{x}{\sin x}.\cos x\right]$$

$$\Rightarrow \frac{du}{dx}=(\sin x)^x(x\cot x+\log\sin x) \qquad \ldots..(2)$$

$v=\sin^{-1}\sqrt{x}$

Differentiating both sides with respect to x , we obtain

$$\frac{dv}{dx}=\frac{1}{\sqrt{1-(\sqrt{x})^2}}.\frac{d}{dx}(\sqrt{x})$$

$$\Rightarrow \frac{dv}{dx}=\frac{1}{\sqrt{1-x}}.\frac{1}{2\sqrt{x}}$$

$$\Rightarrow \frac{dv}{dx}=\frac{1}{2\sqrt{x-x^2}} \qquad \ldots..(3)$$

Therefore, from (1), (2) and (3), we obtain

$$\frac{dy}{dx}=(\sin x)^2(x\cot x+\log\sin x)+\frac{1}{2\sqrt{x-x^2}}$$

Question 9:

Differentiate the function with respect to x.

$x^{\sin x}+(\sin x)^{\cos x}$

Solution 9:

Let $y = x^{\sin x} + (\sin x)^{\cos x}$

Also $u = x^{\sin x}$ and $v = (\sin x)^{\cos x}$

$\therefore y = u + v$

$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ ……(1)

$u = x^{\sin x}$

$\Rightarrow \log u = \log\left(x^{\sin x}\right)$

$\Rightarrow \log u = \sin x \log x$

Differentiating both sides with respect to x, we obtain

$\frac{1}{u}\frac{du}{dx} = \frac{d}{dx}(\sin x).\log x + \sin x.\frac{d}{dx}(\log x)$

$\Rightarrow \frac{du}{dx} = u = \left[\cos x \log x + \sin x \cdot \frac{1}{x}\right]$

$\Rightarrow \frac{du}{dx} == x^{\sin x}\left[\cos x \log x + \frac{\sin x}{x}\right]$ ……(2)

$v = (\sin x)^{\cos x}$

$\Rightarrow \log v = \log(\sin x)^{\cos x}$

$\Rightarrow \log v = cox \log(\sin x)$

Differentiating both sides with respect to x , we obtain

$\frac{1}{v}\frac{dv}{dx} = \frac{d}{dx}(\cos x)\text{x}\log(\text{sinx}) + \cos x\text{x}\frac{d}{dx}\left[\log(\sin x)\right]$

$\Rightarrow \frac{dv}{dx} = v\left[-\sin x.\log(\sin x) + \cos x.\frac{1}{\sin x}.\frac{d}{dx}(\sin x)\right]$

$\Rightarrow \frac{dv}{dx} = (\sin x)^{\cos x}\left[-\sin x \log \sin x + \frac{\cos x}{\sin x}\cos x\right]$

$\Rightarrow \frac{dv}{dx} = (\sin x)^{\cos x}\left[-\sin x \log \sin x + \cot x \cos x\right]$

$\Rightarrow \frac{dv}{dx} = (\sin x)^{\cos x}\left[\cot x \cos x - \sin x \log \sin x\right]$ ….(3)

Therefore, from (1), (2) and (3), we obtain

$\frac{dy}{dx} = x^{\sin x}\left(\cos x \log x + \frac{\sin x}{x}\right) + (\sin x)^{\cos x}\left[\cos x \cot x - \sin x \log \sin x\right]$

Question 10:

Differentiate the function with respect to x.

$$x^{x\cos x}+\frac{x^2+1}{x^2-1}$$

Solution 10:

Let $y=x^{x\cos x}+\dfrac{x^2+1}{x^2-1}$

Also, let $u=x^{x\cos x}$ and $v=\dfrac{x^2+1}{x^2-1}$

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}$$

$$\therefore y=u+v$$

$$u=x^{x\cos x}$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx}=\frac{d}{dx}(x).\cos x\log x+x.\frac{d}{dx}(\cos x).\log x+x\cos x.\frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{du}{dx}=u\left[1.\cos x.\log x+x.(-\sin x)\log x+x\cos x.\frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx}=x^{x\cos x}\left(cox\log x-x\sin x\log x+\cos x\right)$$

$$\Rightarrow \frac{du}{dx}=x^{x\cos x}\left[\cos x(1+\log x)-x\sin x\log x\right] \qquad \text{....(2)}$$

$$v=\frac{x^2+1}{x^2-1}$$

$$\Rightarrow \log v=\log\left(x^2+1\right)-\log\left(x^2-1\right)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}=\frac{dv}{dx}=\frac{2x}{x^2+1}-\frac{2x}{x^2-1}$$

$$\Rightarrow \frac{dv}{dx}=v\left[\frac{2x\left(x^2-1\right)-2x\left(x^2+1\right)}{\left(x^2+1\right)\left(x^2-1\right)}\right]$$

$$\Rightarrow \frac{dv}{dx}=\frac{x^2+1}{x^2-1}\text{x}\left[\frac{\text{-}4x}{\left(x^2+1\right)\left(x^2-1\right)}\right]$$

$$\Rightarrow \frac{dv}{dx}=\frac{-4x}{\left(x^2-1\right)^2} \qquad \text{.......(3)}$$

Therefore, from (1), (2) and (3), we obtain

$$\frac{dy}{dx}=x^{x\cos x}\left[\cos x\left(1+\log x\right)-x\sin x\log x\right]-\frac{4x}{\left(x^2-1\right)^2}$$

Question 11:

Differentiate the function with respect to x.

$$\left(x\cos x\right)^x+\left(x\sin x\right)^{\frac{1}{x}}$$

Solution 11:

Let $y=\left(x\cos x\right)^x+\left(x\sin x\right)^{\frac{1}{x}}$

Also, let $u=\left(x\cos x\right)^x$ and $v=\left(x\sin x\right)^{\frac{1}{x}}$

$\therefore y=u+v$

$$\Rightarrow\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}\qquad\ldots.(1)$$

$u=\left(x\cos x\right)^2$

$\Rightarrow\log u=\log\left(x\cos x\right)^x$

$\Rightarrow\log u=x\log\left(x\cos x\right)$

$\Rightarrow\log u=x\left[\log x+\log\cos x\right]$

$\Rightarrow\log u=x\log x+x\log\cos x$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx}=\frac{d}{dx}(x+\log x)+\frac{d}{dx}(x\log\cos x)$$

$$\Rightarrow\frac{du}{dx}=u\left[\left\{\log x.\frac{d}{dx}\left(x\right)+x.\frac{d}{dx}\left(\log x\right)\right\}+\left\{\log\cos x.\frac{d}{dx}\left(x\right)+x.\frac{d}{dx}\left(\log\cos x\right)\right\}\right]$$

$$\Rightarrow\frac{du}{dx}=(x\cos x)^x\left[\left(\log x.1+x.\frac{1}{x}\right)+\left\{\log\cos x\cdot1+x\cdot\frac{1}{\cos x}\cdot\frac{d}{dx}(\cos x)\right\}\right]$$

$$\Rightarrow\frac{du}{dx}=(x\cos x)^x\left[\left(\log x\cdot1+x\cdot\frac{1}{x}\right)+\left\{\log\cos x\cdot1+x\cdot\frac{1}{\cos x}.\frac{d}{dx}(\cos x)\right\}\right]$$

$$\Rightarrow\frac{du}{dx}=\left(x\cos x\right)^x\left[\left(\log x+1\right)+\left\{\log\cos x+\frac{x}{\cos x}.(-\sin x)\right\}\right]$$

$$\Rightarrow \frac{du}{dx} = (x\cos x)^x \left[(1+\log x) + (\log\cos x - x\tan x)\right]$$

$$\Rightarrow \frac{du}{dx} = (x\cos x)^x \left[1 - x\tan x + (\log x + \log\cos x)\right]$$

$$\Rightarrow \frac{du}{dx} = (x\cos x)^x \left[1 - x\tan x + \log(x\cos x)\right] \qquad \ldots\ldots(2)$$

$$v = (x\sin x)^{\frac{1}{x}}$$

$$\Rightarrow \log v = \log(x\sin x)^{\frac{1}{x}}$$

$$\Rightarrow \log v = \frac{1}{x}\log(x\sin x)$$

$$\Rightarrow \log v = \frac{1}{x}(\log x + \log\sin x)$$

$$\Rightarrow \log v = \frac{1}{x}\log x + \frac{1}{x}\log\sin x$$

Differentiating both sides with respect to x , we obtain

$$\frac{1}{v}\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{x}\log x\right) + \frac{d}{dx}\left[\frac{1}{x}\log(\sin x)\right]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \left[\log x.\frac{d}{dx}\left(\frac{1}{x}\right) + \frac{1}{x}.\frac{d}{dx}(\log x)\right] + \left[\log(\sin x).\frac{d}{dx}\left(\frac{1}{x}\right) + \frac{1}{x}.\frac{d}{dx}\{\log(\sin x)\}\right]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \left[\log x.\left(-\frac{1}{x^2}\right) + \frac{1}{x}.\frac{1}{x}\right] + \left[\log(\sin x).\left(-\frac{1}{x^2}\right) + \frac{1}{x}.\frac{1}{\sin x}.\frac{d}{dx}(\sin x)\right]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \frac{1}{x^2}(1-\log x) + \left[-\frac{\log(\sin x)}{x^2} + \frac{1}{x\sin x}.\cos x\right]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \frac{1}{x^2}(x\sin x)^{\frac{1}{x}} + \left[\frac{1-\log x}{x^2} + \frac{-\log(\sin x) + x\cot x}{x^2}\right]$$

$$\Rightarrow \frac{dv}{dx} = (x\sin x)^{\frac{1}{x}}\left[\frac{1-\log x - \log(\sin x) + x\cot x}{x^2}\right]$$

$$\Rightarrow \frac{dv}{dx} = (x\sin x)^{\frac{1}{x}}\left[\frac{1-\log(x\sin x) + x\cot x}{x^2}\right] \qquad \ldots\ldots\ldots(3)$$

Therefore, from (1), (2) and (3), we obtain

$$\frac{dy}{dx} = (x\cos x)^2\left[1 - x\tan x + \log(x\cos x)\right] + (x\sin x)^{\frac{1}{x}}\left[\frac{x\cot x + 1 - \log(x\sin x)}{x^2}\right]$$

Question 12:

Find $\frac{dy}{dx}$ of function.

$x^y + y^x = 1$

Solution 12:

The given function is $x^y + y^x = 1$

Let $x^y = u$ and $y^x = v$

Then, the function becomes $u + v = 1$

$$\therefore \frac{du}{dx} + \frac{dv}{dx} = 0 \qquad \text{.........(1)}$$

$u = x^y$

$\Rightarrow \log u = \log\left(x^y\right)$

$\Rightarrow \log u = y \log x$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx} = \log x \frac{dy}{dx} + y.\frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{du}{dx} = u\left[\log x \frac{dy}{dx} + y.\frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx} = x^y\left(\log x \frac{dy}{dx} + \frac{y}{x}\right)$$

$$\text{.......(2)}$$

$v = y^x$

$\Rightarrow \log v = \log(y^x)$

$\Rightarrow \log v = x \log y$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}\cdot\frac{dv}{dx} = \log y \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\log y)$$

$$\Rightarrow \frac{dv}{dx} = v\left(\log y \cdot 1 + x \cdot \frac{1}{y} \cdot \frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dv}{dx} = y^x\left(\log y + \frac{x}{y}\frac{dy}{dx}\right)$$

$$\text{.....(3)}$$

Therefore, from (1), (2) and (3), we obtain

$$x^y\left(\log x\frac{dy}{dx}+\frac{y}{x}\right)+y^x\left(\log y+\frac{x}{y}\frac{dy}{dx}\right)=0$$

$$\Rightarrow\left(x^2+\log x+xy^{y-1}\right)\frac{dy}{dx}=-\left(yx^{y-1}+y^x\log y\right)$$

$$\therefore\frac{dy}{dx}=-\frac{yx^{y-1}+y^x\log y}{x^y\log x+xy^{x-1}}$$

Question 13:

Find $\dfrac{dy}{dx}$ of function $y^x=x^y$

Solution 13:

The given function is $y^x=x^y$

Taking logarithm on both sides, we obtain.

$x\log y=y\log x$

Differentiating both sides with respect to x, we obtain

$$\log y.\frac{d}{dx}(x)+x.\frac{d}{dx}(\log y)=\log x.\frac{d}{dx}(y)+y.\frac{d}{dx}(\log x)$$

$$\Rightarrow\log y.1+x.\frac{1}{y}.\frac{dy}{dx}=\log x.\frac{dy}{dx}+y.\frac{1}{x}$$

$$\Rightarrow\log y+\frac{x}{y}\frac{dy}{dx}=\log x\frac{dy}{dx}+\frac{y}{x}$$

$$\Rightarrow\left(\frac{x}{y}-\log x\right)\frac{dy}{dx}=\frac{y}{x}-\log y$$

$$\Rightarrow\left(\frac{x-y\log x}{y}\right)\frac{dy}{dx}=\frac{y-x\log y}{x}$$

$$\Rightarrow\left(\frac{x-y\log x}{y}\right)\frac{dy}{dx}=\frac{y-x\log y}{x}$$

$$\therefore\frac{dy}{dx}=\frac{y}{x}\left(\frac{y-x\log y}{x-y\log x}\right)$$

Question 14:

Find $\dfrac{dy}{dx}$ of function $(\cos x)^y = (\cos y)^x$

Solution 14:

The given function is $(\cos x)^y = (\cos y)^x$

Taking logarithm on both sides, we obtain.

$y = \log\cos x = x\log\cos y$

Differentiating both sides with respect to x , we obtain

$$\log\cos x\cdot\frac{dy}{dx} + y\cdot\frac{d}{dx}(\log\cos x) = \log\cos y\cdot\frac{d}{dx}(x) + x\cdot\frac{d}{dx}(\log\cos y)$$

$$\Rightarrow \log\cos x.\frac{dy}{dx} + y.\frac{1}{\cos x}.\frac{d}{dx}(\cos x) = \log\cos y.1 + x.\frac{1}{\cos y}.\frac{d}{dx}(\cos y)$$

$$\Rightarrow \log\cos x\frac{dy}{dx} + \frac{y}{\cos x}.(-\sin x) = \log\cos y + \frac{x}{\cos y}(-\sin y).\frac{dy}{dx}$$

$$\Rightarrow \log\cos x\frac{dy}{dx} - y\tan x = \log\cos y - x\tan y\frac{dy}{dx}$$

$$\Rightarrow (\log\cos x + x\tan y)\frac{dy}{dx} = y\tan x + \log\cos y$$

$$\therefore \frac{dy}{dx} = \frac{y\tan x + \log\cos y}{x\tan y + \log\cos x}$$

Question 15:

Find $\dfrac{dy}{dx}$ of function $xy = e^{(x-y)}$

Solution 15:

The given function is $xy = e^{(x-y)}$

Taking logarithm on both sides, we obtain.

$$\log(xy) = \log\left(e^{x-y}\right)$$

$$\Rightarrow \log x + \log y = (x-y)\log e$$

$$\Rightarrow \log x + \log y = (x-y)\times 1$$

$$\Rightarrow \log x + \log y = x - y$$

Differentiating both sides with respect to x , we obtain

$$\frac{d}{dx}(\log x)+\frac{d}{dx}(\log y)=\frac{d}{dx}(x)-\frac{dy}{dx}$$

$$\Rightarrow \frac{1}{x}+\frac{1}{y}\frac{dy}{dx}=1-\frac{1}{x}$$

$$\Rightarrow \left(1+\frac{1}{y}\right)\frac{dy}{dx}=\frac{x-1}{x}$$

$$\therefore \frac{dy}{dx}=\frac{y(x-1)}{x(y+1)}$$

Question 16:

Find the derivative of the function given by $f(x)=(1-x)(1+x^2)(1+x^4)(1+x^8)$ and hence find $f^{'}(1)$

Solution 16:

The given relationship is $f(x)=(1-x)(1+x^2)(1+x^4)(1+x^8)$

Taking logarithm on both sides, we obtain.

$$\log f(x)=\log(1+x)+\log(1+x^2)+\log(1+x^4)+\log(1+x^8)$$

Differentiating both sides with respect to x , we obtain

$$\frac{1}{f(x)}\cdot\frac{d}{dx}[f(x)]=\frac{d}{dx}\log(1+x)+\frac{d}{dx}\log(1+x^2)+\frac{d}{dx}\log(1+x^4)+\frac{d}{dx}\log(1+x^8)$$

$$\Rightarrow \frac{1}{f(x)}\cdot f^{'}(x)=\frac{1}{1+x}\cdot\frac{d}{dx}(1+x)+\frac{1}{1+x^2}\cdot\frac{d}{dx}(1+x^2)+\frac{1}{1+x^4}\cdot\frac{d}{dx}(1+x^4)+\frac{1}{1+x^8}\cdot\frac{d}{dx}(1+x^8)$$

$$\Rightarrow f^{'}(x)=f(x)\left[\frac{1}{1+x}+\frac{1}{1+x^2}.2x+\frac{1}{1+x^4}.4x^3+\frac{1}{1+x^8}.8x^7\right]$$

$$\therefore f^{'}(x)=(1+x)(1+x^2)(1+x^4)(1+x^8)\left[\frac{1}{1+x}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\frac{8x^7}{1+x^8}\right]$$

Hence, $f^{'}(1)=(1+1)(1+1^2)(1+1^4)(1+1^8)\left[\frac{1}{1+1}+\frac{2\text{x}1}{1+1^2}+\frac{4\text{x}1^3}{1+1^4}+\frac{8\text{x}1^7}{1+1^8}\right]$

$$=2x2x2x2\left[\frac{1}{2}+\frac{2}{2}+\frac{4}{2}+\frac{8}{2}\right]$$

$$=16x\left(\frac{1+2+4+8}{2}\right)$$

$$=16x\frac{15}{2}=120$$

Question 17:

Differentiate $\left(x^2-5x+8\right)\left(x^3+7x+9\right)$ in three ways mentioned below

- **i.** By using product rule.
- **ii.** By expanding the product to obtain a single polynomial
- **iii.** By logarithm Differentiate

Do they all given the same answer?

Solution 17:

Let $y=\left(x^2-5x+8\right)\left(x^3+7x+9\right)$

(i) Let $x=x^2-5x+8$ and $u=x^3+7x+9$

$\therefore y=uv$

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dv}.v+u.\frac{dv}{dx} \qquad \left(\text{By using product rule}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{d}{dx}\left(x^2-5x+8\right).\left(x^3+7x+9\right)+\left(x^2-5x+8\right).\frac{d}{dx}\left(x^3+7x+9\right)$$

$$\Rightarrow \frac{dy}{dx}=\left(2x-5\right)\left(x^3+7x+9\right)+\left(x^2-5x+8\right)\left(3x^2+7\right)$$

$$\Rightarrow \frac{dy}{dx}=2x\left(x^3+7x+9\right)-5\left(x^3+7x+9\right)+x^2\left(3x^2+7\right)-5x\left(3x^2+7\right)-8\left(3x^2+7\right)$$

$$\Rightarrow \frac{dy}{dx}=\left(2x^4+14x^2+18x\right)-5x^3-35x-45+\left(3x^4+7x^2\right)-15x^3-35x+24x^2+56$$

$$\therefore \frac{dy}{dx}=5x^4-20x^3+45x^2-52x+11$$

(ii)

$y=\left(x^2-5x+8\right)\left(x^3+7x+9\right)$

$=x^2\left(x^3+7x+9\right)-5x\left(x^3+7x+9\right)+8\left(x^3+7x+9\right)$

$=x^5+7x^3+9x^2-5x^4-35x^2-45x+8x^3+56x+72$

$=x^5-5x^4+15x^3-26x^2+11x+72$

$\therefore \frac{dy}{dx}=\frac{d}{dx}\left(x^5-5x^4+15x^3-26x^2+11x+72\right)$

$=\frac{d}{dx}\left(x^5\right)-5\frac{d}{dx}\left(x^4\right)+15\frac{d}{dx}\left(x^3\right)-26\frac{d}{dx}\left(x^2\right)+11\frac{d}{dx}(x)+\frac{d}{dx}(72)$

$=5x^4-5 \text{ x } 4x^3+15 \text{ x } 3x^2-26 \text{ x } 2x+11 \text{ x } 1+0$

$=5x^4-20x^3+45x^2-52x+11$

(iii) Taking logarithm on both sides, we obtain.

$\log y=\log\left(x^2-5x+8\right)+\log\left(x^3+7x+9\right)$

Differentiating both sides with respect to x , we obtain

$\frac{1}{y}\frac{dy}{dx}=\frac{d}{dx}\log\left(x^2-5x+8\right)+\frac{d}{dx}\log\left(x^3+7x+9\right)$

$\Rightarrow \frac{1}{y}\frac{dy}{dx}=\frac{1}{x^2-5x+8}.\frac{d}{dx}\left(x^2-5x+8\right)+\frac{1}{x^3+7x+9}.\frac{d}{dx}\left(x^3+7x+9\right)$

$\Rightarrow \frac{dy}{dx}=y\left[\frac{1}{x^2-5x+8}\text{ x}\left(2x\text{-}5\right)+\frac{1}{x^3+7x+9}\text{ x}\left(3x^2+7\right)\right]$

$\Rightarrow \frac{dy}{dx}=\left(x^2-5x+8\right)\left(x^3+7x+9\right)\left[\frac{2x-5}{x^3-5x+8}+\frac{3x^2+7}{x^3+7x+9}\right]$

$\Rightarrow \frac{dy}{dx}=\left(x^2-5x+8\right)\left(x^3+7x+9\right)\left[\frac{(2x-5)(x^3+7x+9)+\left(3x^2+7\right)\left(x^2-5x+8\right)}{\left(x^2-5x+8\right)+\left(x^3+7x+9\right)}\right]$

$\Rightarrow \frac{dy}{dx}=2x\left(x^3+7x+9\right)-5\left(x^3+7x+9\right)+3x^2\left(x^2-5x+8\right)+7\left(x^2-5x+8\right)$

$\Rightarrow \frac{dy}{dx}=\left(2x^4+14x^2+18x\right)-5x^3-35x-45+\left(3x^4-15x^3+24x^2\right)+\left(7x^2-35x+56\right)$

$\Rightarrow \frac{dy}{dx}=5x^2-20x^3+45x^2-52x+11$

From the above three observations, it can be concluded that all the result of $\frac{dy}{dx}$ are same.

Question 18:

If u, v and w are functions of x, then show that $\frac{d}{dx}(u.v.w) = \frac{du}{dx}v.w + u\frac{dv}{dx}.w + u.v\frac{dw}{dx}$

In two ways-first by repeated application of product rule, second by logarithmic differentiation.

Solution 18:

Let $y = u.v.w = u.(v.w)$

By applying product rule, we obtain

$$\frac{dy}{dx} = \frac{du}{dx}.(v.w) + u.\frac{d}{dx}(v.w) \qquad \text{(Again applying product rule)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx}v.w + u\left[\frac{dv}{dx}.w + v.\frac{dv}{dx}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx}v.w + u.\frac{dv}{dx}.w + u.v\frac{dw}{dx}$$

By taking logarithm on both sides of the equation $y = u.v.w$, we obtain

$$\log y = \log u + \log v + \log w$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}.\frac{dy}{dx} = \frac{d}{dx}(\log u) + \frac{d}{dx}(\log v) + \frac{d}{dx}(\log w)$$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}$$

$$\Rightarrow \frac{dy}{dx} = y\left(\frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx} = u.v.w\left(\frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}\right)$$

$$\therefore \frac{dy}{dx} = \frac{du}{dx}.v.w + u.\frac{dv}{dx}.w. + u.v.\frac{dw}{dx}$$

Exercise 5.6

Question 1:

If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$

$x = 2at^2,\ y = at^4$

Solution 1:

The given equations are $x = 2at^2$ and $y = at^4$

Then,

$$\frac{dx}{dt} = \frac{d}{dt}(2at^2) = 2a.\frac{d}{dt}(t^2) = 2a.2t = 4at$$

$$\frac{dy}{dx} = \frac{d}{dt}(at^4)a \cdot \frac{d}{dt}(t^4) = a \cdot 4 \cdot t^3 = 4at^3$$

$$\therefore \frac{dy}{dt} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{4at^3}{4at} = t^2$$

Question 2:

If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$

$x = a\cos\theta,\ y = b\cos\theta$

Solution 2:

The given equations are $x = a\cos\theta$ and $y = b\cos\theta$

Then, $\frac{dx}{d\theta} = \frac{d}{d\theta}\left(a\cos\theta\right) = a(-\sin\theta) = -a\sin\theta$

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(b\cos\theta) = b(-\sin\theta) = -b\sin\theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{-b\sin\theta}{-a\sin\theta} = \frac{b}{a}$$

Question 3:

If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$

$x = \sin t,\ y = \cos 2t$

Solution 3:

The given equations are $x = \sin t$ and $y = \cos 2t$

Then, $\frac{dx}{dt} = \frac{d}{dt}(\sin t) = \cos t$

$$\frac{dy}{dt} = \frac{d}{dt}(\cos 2t) = -\sin 2t \cdot \frac{d}{dt}(2t) = -2\sin 2t$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dx}\right)}{\left(\frac{dx}{dt}\right)} = \frac{-2\sin 2t}{\cos t} = \frac{-2 \cdot 2\sin t \cos t}{\cos t} = -4\sin t$$

Question 4:

If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$

$x = 4t,\ y = \frac{4}{t}$

Solution 4:

The equations are $x = 4t$ and $y = \frac{4}{t}$

$$\frac{dx}{dt} = \frac{d}{dt}(4t) = 4$$

$$\frac{dy}{dt} = \frac{d}{dt}\left(\frac{4}{t}\right) = 4 \cdot \frac{d}{dt}\left(\frac{1}{t}\right) = 4 \cdot \left(\frac{-1}{t^2}\right) = \frac{-4}{t^2}$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\left(\frac{-4}{t^2}\right)}{4} = \frac{-1}{t^2}$$

Question 5:

If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$

$x = \cos\theta - \cos 2\theta,\ y = \sin\theta - \sin 2\theta$

Solution 5:

The given equations are $x = \cos\theta - \cos 2\theta$ and $y = \sin\theta - \sin 2\theta$

Then, $\frac{dx}{d\theta} = \frac{d}{d\theta}(\cos\theta - \cos 2\theta) = \frac{d}{d\theta}(\cos\theta) - \frac{d}{d\theta}(\cos 2\theta)$

$= -\sin\theta(-2\sin 2\theta) = 2\sin 2\theta - \sin\theta$

$\frac{dy}{d\theta} = \frac{d}{d\theta}(\sin\theta - \sin 2\theta) = \frac{d}{d\theta}(\sin\theta) - \frac{d}{d\theta}(\sin 2\theta)$

$= \cos\theta - 2\cos\theta$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{\cos\theta - 2\cos\theta}{2\sin 2\theta - \sin\theta}$$

Question 6:

If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$

$x = a(\theta - \sin\theta),\ y = a(1 + \cos\theta)$

Solution 6:

The given equations are $x = a(\theta - \sin\theta)$ and $y = a(1 + \cos\theta)$

Then, $\frac{dx}{d\theta} = a\left[\frac{d}{d\theta}(\theta) - \frac{d}{d\theta}(\sin\theta)\right] = a(1 - \cos\theta)$

$$\frac{dy}{d\theta}=a\left[\frac{d}{d\theta}(1)+\frac{d}{d\theta}(\cos\theta)\right]=a[0+(-\sin\theta)]=-a\sin\theta$$

$$\therefore \frac{dy}{dx}=\frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}=\frac{-a\sin\theta}{a(1-\cos\theta)}=\frac{-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}=\frac{-\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}=-\cot\frac{\theta}{2}$$

Question 7:

If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$

$$x=\frac{\sin^3 t}{\sqrt{\operatorname{co}x2t}},\ y=\frac{\cos^3 t}{\sqrt{\cos 2t}}$$

Solution 7:

The given equations are $x=\frac{\sin^3 t}{\sqrt{\cos 2t}}$ and $y=\frac{\cos^3 t}{\sqrt{\cos 2t}}$

Then, $\frac{dx}{dt}=\frac{d}{dt}\left[\frac{\sin^3 t}{\sqrt{\cos 2t}}\right]$

$$=\frac{\sqrt{\cos 2t}-\frac{d}{dt}(\sin^3 t)-\sin^3 t\cdot\frac{d}{dt}\sqrt{\cos 2t}}{\cos 2t}$$

$$=\frac{\sqrt{\cos 2t}\cdot 3\sin^2 t\cdot\frac{d}{dt}(\sin t)-\sin^3 t\ x\frac{1}{2\sqrt{\cos 2t}}\cdot\frac{d}{dt}(\cos 2t)}{\cos 2t}$$

$$=\frac{3\sqrt{\cos 2t}\cdot\sin^2 t\cos t-\frac{\sin^3 t}{2\sqrt{\cos 2t}}\cdot(-2\sin 2t)}{\cos 2t\sqrt{\cos 2t}}$$

$$=\frac{3\cos 2t\sin^2 t\cot t+\sin^2 t\sin 2t}{\cos 2t\sqrt{\cos 2t}}$$

$$\frac{dy}{dt}=\frac{d}{dt}\left[\frac{\cos^3 t}{\sqrt{\cos 2t}}\right]$$

$$= \frac{\sqrt{\cos 2t} \cdot \frac{d}{dt}(\cos^3 t) - \cos^3 t \cdot \frac{d}{dt}\left(\sqrt{\cos 2t}\right)}{\cos 2t}$$

$$= \frac{\sqrt{\cos 2t}\, 3\cos^2 t \cdot \frac{d}{dt}(\cos t) - \cos^3 t \cdot \frac{1}{2\sqrt{\cos 2t}} \cdot \frac{d}{dt}(\cos 2t)}{\cos 2t}$$

$$= \frac{3\sqrt{\cos 2t}\cos^2 t(-\sin t) - \cos^3 t.\frac{1}{\sqrt{\cos 2t}} \cdot (-2\sin 2t)}{\cos 2t}$$

$$= \frac{-3\cos 2t.\cos^2 t.\sin t + \cos^3 t \sin 2t}{\cos 2t \cdot \sqrt{\cos 2t}}$$

$$\therefore \frac{dy}{dx}\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{-3\cos 2t \cdot \cos^2 t + \cos^3 t \sin 2t}{3\cos 2t \sin^2 t \cos t + \sin^3 t \sin 2t}$$

$$= \frac{3\cos 2t.\cos^2 t \sin t + \cos^3 t(2\sin t \cos t)}{3\cos 2t \sin^2 \cdot \cos t + \sin^3 t(2\sin t \cos t)}$$

$$= \frac{\sin t \cos t[-3\cos 2t \cdot \cos t + 2\cos^3 t]}{\sin t \cos t[3\cos 2t \sin t + 2\sin^3 t]}$$

$$= \frac{\left[-3(2\cos^2 t - 1)\cos t + 2\cos^3 t\right]}{\left[3(1-2\sin^2 t)\sin t + 2\sin^3 t\right]} \qquad \begin{bmatrix} \cos 2t = (2\cos^2 t - 1) \\ \cos 2t = (1 - 2\sin^2 t) \end{bmatrix}$$

$$\frac{-4\cos^3 t + 3\cos t}{3\sin t - 4\sin^3 t} \qquad \begin{bmatrix} \cos 3t = 4\cos^3 t - 3\cos t \\ \sin 3t = 3\sin t - 4\sin^2 t \end{bmatrix}$$

$$= \frac{-\cos 3t}{\sin 3t}$$

$$= -\cot 3t$$

Question 8:

If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$

$$x = a\left(\cos t + \log \tan \frac{t}{2}\right),\ y = a \sin t$$

Solution 8:

The given equations are $x = a\left(\cos t + \log\tan\frac{t}{2}\right)$ and $y = a\sin t$

Then, $\frac{dx}{dt} = a\cdot\left[\frac{d}{dt}(\cos t) + \frac{d}{dt}\left(\log\tan\frac{t}{2}\right)\right]$

$$= a\left[-\sin t + \frac{1}{\tan\frac{t}{2}}\cdot\frac{d}{dt}\left(\tan\frac{t}{2}\right)\right]$$

$$= a\left[-\sin t + \cot\frac{t}{2}\cdot\sec^2\frac{t}{2}\cdot\frac{d}{dt}\left(\frac{t}{2}\right)\right]$$

$$= a\left[-\sin t + \frac{\cos\frac{t}{2}}{\sin\frac{t}{2}}\times\frac{1}{\cos^2\frac{t}{2}}\times\frac{1}{2}\right]$$

$$= a\left[-\sin t + \frac{1}{2\sin\frac{t}{2}\cos\frac{t}{2}}\right]$$

$$= a\left(-\sin t + \frac{1}{\sin t}\right)$$

$$= a\left(\frac{-\sin^2 t + 1}{\sin t}\right)$$

$$= a\frac{\cos^2 t}{\sin t}$$

$$\frac{dy}{dt} = a\frac{d}{dt}(\sin t) = a\cos t$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{a\cos t}{\left(a\frac{\cos^2 t}{\sin t}\right)} = \frac{\sin t}{\cos t} = \tan t$$

Question 9:

If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$

$x = a\sec,\ y = b\tan\theta$

Solution 9:

The given equations are $x = a\sec$ and $y = b\tan\theta$

Then, $\frac{dx}{d\theta} = a \cdot \frac{d}{d\theta}(\sec\theta) = a\sec\theta\tan\theta$

$\frac{dy}{d\theta} = b \cdot \frac{d}{d\theta}(\tan\theta) = b\sec^2\theta$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{b\sec^2\theta}{a\sec\theta\tan\theta} = \frac{b}{a}\sec\theta\cot\theta = \frac{b\cos\theta}{a\cos\theta\sin\theta} = \frac{b}{a} \times \frac{1}{\sin\theta} = \frac{b}{a}\operatorname{cosec}\theta$$

Question 10:

If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$

$x = a(\cos\theta + \theta\sin\theta),\ y = a(\sin\theta - \theta\cos\theta)$

Solution 10:

The given equations are $x = a(\cos\theta + \theta\sin\theta)$ and $y = a(\sin\theta - \theta\cos\theta)$

Then, $\frac{dx}{d\theta} = a\left[\frac{d}{d\theta}\cos\theta + \frac{d}{d\theta}(\theta\sin\theta)\right] = a\left[-\sin\theta + \theta\frac{d}{d\theta}(\sin\theta) + \sin\theta\frac{d}{d\theta}(\theta)\right]$

$= a[-\sin\theta + \theta\cos\theta + \sin\theta] = a\theta\cos\theta$

$\frac{dx}{d\theta} = a\left[\frac{d}{d\theta}(\sin\theta) - \frac{d}{d\theta}(\theta\cos\theta)\right] = a\left[\cos\theta - \left\{\theta\frac{d}{d\theta}(\cos\theta) + \cos\theta \cdot \frac{d}{d\theta}(\theta)\right\}\right]$

$= a[\cos\theta + \theta\sin\theta - \cos\theta]$

$= a\theta\sin\theta$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{a\theta\sin\theta}{a\theta\sin\theta} = \tan\theta$$

Question 11:

If $x=\sqrt{a^{\sin -1 t}}$, $y=\sqrt{a^{\cos -1 t}}$, show that $\frac{dy}{dx}=-\frac{y}{x}$

Solution 11:

The given equations are $x=\sqrt{a^{\sin -1 t}}$ and $y=\sqrt{a^{\cos -1 t}}$

$x=\sqrt{a^{\sin -1 t}}$ and $y=\sqrt{a^{\cos -1 t}}$

$\Rightarrow x=\left(a^{\sin -1 t}\right)$ and $y=\left(a^{\cos -1 t}\right)^{\frac{1}{2}}$

$\Rightarrow x=a^{\frac{1}{2}\sin -1 t}$ and $y=a^{\frac{1}{2}\cos -1 t}$

Consider $x=a^{\frac{1}{2}\sin -1 t}$

Taking logarithm on both sides, we obtain.

$\log x=\frac{1}{2}\sin^{-1} t\log a$

$\therefore \frac{1}{x}\cdot\frac{dx}{dt}=\frac{1}{2}\log a\cdot\frac{d}{dt}(\sin^{-1} t)$

$\Rightarrow \frac{dx}{dt}=\frac{x}{2}\log a\cdot\frac{1}{\sqrt{1-t^2}}$

$\Rightarrow \frac{dx}{dt}=\frac{x\log a}{2\sqrt{1-t^2}}$

Then, consider

$y=a^{\frac{1}{2}\cos^{-1} t}$

Taking logarithm on both sides, we obtain.

$\log y=\frac{1}{2}\cos^{-1} t\log a$

$$\therefore \frac{1}{y}\cdot\frac{dy}{dx}=\frac{1}{2}\log a\cdot\frac{d}{dt}\left(\cos^{-1}t\right)$$

$$\Rightarrow \frac{dy}{dt}=\frac{y\log a}{2}\cdot\left(\frac{-1}{\sqrt{1-t^2}}\right)$$

$$\Rightarrow \frac{dy}{dt}=\frac{-y\log a}{2\sqrt{1-t^2}}$$

$$\therefore \frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{\left(\frac{-y\log a}{2\sqrt{1-t^2}}\right)}{\left(\frac{x\log a}{2\sqrt{1-t^2}}\right)}=-\frac{y}{x}$$

Hence proved.

Exercise 5.7

Question 1:
Find the second order derivatives of the function. x^2+3x+2

Solution 1:
Let $y=x^2+3x+2$
Then,

$$\frac{dy}{dx}=\frac{d}{dx}(x^2)+\frac{d}{dx}(3x)+\frac{d}{dx}(2)=2x+3+0=2x+3$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}(2x+3)=\frac{d}{dx}(2x)+\frac{d}{dx}(3)=2+0=2$$

Question 2:
Find the second order derivatives of the function. x^{20}

Solution 2:
Let $y=x^{20}$
Then,

$$\frac{dy}{dx}=\frac{d}{dx}(x^{20})=20x^{19}$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}(20x^{19})=20\frac{d}{dx}(x^{19})=20\cdot 19\cdot x^{18}=380x^{18}$$

Question 3:

Find the second order derivatives of the function. $x\cdot \cos x$

Solution 3:

Let $y=x\cdot\cos x$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}\left(x\cdot\cos x\right)=\cos x\cdot\frac{d}{dx}(x)+x\frac{d}{dx}(\cos x)=\cos x\cdot 1+x(-\sin x)=\cos x-x\sin x$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}\left[\cos x-\sin x\right]=\frac{d}{dx}(\cos x)-\frac{d}{dx}(x\sin x)$$

$$=-\sin x-\left[\sin x\cdot\frac{d}{dx}(x)+x\cdot\frac{d}{dx}(\sin x)\right]$$

$$=-\sin x-(\sin x+\cos x)$$

$$=-(x\cos x+2\sin x)$$

Question 4:

Find the second order derivatives of the function. $\log x$

Solution 4:

Let $y=\log x$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}(\log x)=\frac{1}{x}$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{1}{x}\right)=\frac{-1}{x^2}$$

Question 5:

Find the second order derivatives of the function. $x^3 \log x$

Solution 5:

Let $y = x^3 \log x$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}\left[x^3 \log x\right] = \log x \cdot \frac{d}{dx}(x^3) + x^3 \cdot \frac{d}{dx}(\log x)$$

$$= \log x \cdot 3x^2 + x^3 \cdot \frac{1}{x} = \log x \cdot 3x^2 + x^2$$

$$= x^2(1 + 3\log x)$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left[x^2(1 + 3\log x)\right]$$

$$= (1 + 3\log x) \cdot \frac{d}{dx}(x^2) + x^2 \frac{d}{dx}(1 + 3\log x)$$

$$= (1 + 3\log x) \cdot 2x + x^3 \cdot \frac{3}{x}$$

$$= 2x + 6\log x + 3x$$

$$= 5x + 6x\log x$$

$$= x(5 + 6\log x)$$

Question 6:

Find the second order derivatives of the function. $e^x \sin 5x$

Solution 6:

Let $y = e^x \sin 5x$

$$\frac{dy}{dx} = \frac{d}{dx}(e^x \sin 5x) = \sin 5x \frac{d}{dx}(e^x) + e^x \frac{d}{dx}(\sin 5x)$$

$$= \sin 5x \cdot e^x + e^x \cdot \cos 5x \cdot \frac{d}{dx}(5x) = e^x \sin 5x + e^x \cos 5x \cdot 5$$

$$= e^x(\sin 5x + 5\cos 5x)$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left[e^x(\sin 5x + 5\cos 5x)\right]$$

$$= (\sin 5x + 5\cos 5x) \cdot \frac{d}{dx}(e^x) + e^x . \frac{d}{dx}(\sin 5x + 5\cos 5x)$$

$$= (\sin 5x + 5\cos 5x)e^x + e^x \left[\cos 5x \cdot \frac{d}{dx}(5x) + 5(-\sin 5x) \cdot \frac{d}{dx}(5x)\right]$$

$$= e^x(\sin 5x + 5\cos 5x) + e^x(5\cos 5x - 25\sin 5x)$$

Then, $e^x(10\cos 5x - 24\sin 5x) = 2e^x(5\cos 5x - 12\sin 5x)$

Question 7:

Find the second order derivatives of the function. $e^{6x}\cos 3x$

Solution 7:

Let $y = e^{6x}\cos 3x$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}(e^{6x}\cos 3x) = \cos 3x \cdot \frac{d}{dx}(e^{6x}) + e^{6x} \cdot \frac{d}{dx}(\cos 3x)$$

$$= \cos 3x \cdot e^{6x} \cdot \frac{d}{dx}(6x) + e^{6x} \cdot (-\sin 3x) \cdot \frac{d}{dx}(3x)$$

$$= 6e^{6x}\cos 3x - 3e^{6x}\sin 3x \ldots\ldots(1)$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}(6e^{6x}\cos 3x - 3e^{6x}\sin 3x) = 6 \cdot \frac{d}{dx}(e^{6x}\cos 3x) - 3 \cdot \frac{d}{dx}(e^{6x}\sin 3x)$$

$$= 6 \cdot \left[6e^{6x}\cos 3x - 3e^{6x}\sin 3x\right] - 3 \cdot \left[\sin 3x \cdot \frac{d}{dx}(e^{6x}) + e^{6x} \cdot \frac{d}{dx}(\sin 3x)\right] \qquad \text{[using (1)]}$$

$$= 36e^{6x}\cos 3x - 18e^{6x}\sin 3x - 3\left[\sin 3x \cdot e^{6x} \cdot 6 + e^{6x} \cdot \cos 3x - 3\right]$$

$$= 36e^{6x}\cos 3x - 18e^{6x}\sin 3x - 18e^{6x}\sin 3x - 9e^{6x}\cos 3x$$

$$= 27e^{6x}\cos 3x - 36e^{6x}\sin 3x$$

$$= 9e^{6x}(3\cos 3x - 4\sin 3x)$$

Question 8:

Find the second order derivatives of the function. $\tan^{-1} x$

Solution 8:

Let $y = \tan^{-1} x$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{1}{1+x^2}\right)=\frac{d}{dx}(1+x^2)^{-1}=(-1)\cdot(1+x^2)^{-2}\cdot\frac{d}{dx}(1+x^2)-\frac{1}{\left(1+x^2\right)^2}\times 2x=-\frac{2x}{(1+x^2)^2}$$

Question 9:

Find the second order derivatives of the function. $\log(\log x)$

Solution 9:

Let $y=\log(\log x)$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}[\log(\log x)]=\frac{1}{\log x}\cdot\frac{d}{dx}(\log x)=\frac{1}{\log x}=(x\log x)^{-1}$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}\left[(x\log x)^{-1}\right]=(-1)\cdot(x\log x)^{-2}\frac{d}{dx}(x\log x)$$

$$=\frac{-1}{(x\log x)^2}\cdot\left[\log x\cdot\frac{d}{dx}(x)+x\cdot\frac{d}{dx}(\log x)\right]$$

$$=\frac{-1}{(x\log x)^2}\cdot\left[\log x\cdot 1x\cdot\frac{1}{x}\right]=\frac{-1(1+\log x)}{(x\log x)^2}$$

Question 10:

Find the second order derivatives of the function. $\sin(\log x)$

Solution 10:

Let $y=\sin(\log x)$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}\left[\sin(\log x)\right]=\cos(\log x)\cdot\frac{d}{dx}(\log x)=\frac{\cos(\log x)}{x}$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}\left[\frac{\cos(\log x)}{x}\right]$$

$$= \frac{x \cdot \frac{d}{dx}[\cos(\log x)] - \cos(\log x) \cdot \frac{d}{dx}(x)}{x^2}$$

$$= \frac{x\left[-\sin(\log x) \cdot \frac{d}{dx}(\log x)\right] - \cos(\log x) \cdot 1}{x^2}$$

$$= \frac{-x\sin(\log x) \cdot \frac{1}{x} - \cos(\log x)}{x^2}$$

$$= \frac{-[\sin(\log x) + \cos(\log x)]}{x^2}$$

Question 11:

If $y = 5\cos x - 3\sin x$, prove that $\frac{d^2y}{dx^2} + y = 0$

Solution 11:

It is given that, $y = 5\cos x - 3\sin x$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}(5\cos x) - \frac{d}{dx}(3\sin x) = 5\frac{d}{dx}(\cos x) - 3\frac{d}{dx}(\sin x)$$

$$= 5(-\sin x) - 3\cos x = -(5\sin x + 3\cos x)$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}[-(5\sin x + 3\cos x)]$$

$$= -\left[5 \cdot \frac{d}{dx}(\sin x) + 3 \cdot \frac{d}{dx}(\cos x)\right]$$

$$= [5\cos x + 3(-\sin x)]$$

$$= -[5\cos x - 3\sin x]$$

$$= -y$$

$$\therefore \frac{d^2y}{dx^2} + y = 0$$

Hence, proved.

Question 12:

If $y=\cos^{-1}x$, find $\dfrac{d^2y}{dx^2}$ in terms of y alone.

Solution 12:

It is given that, $y=\cos^{-1}x$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}(\cos^{-1}x)=\frac{-1}{\sqrt{1-x^2}}=-(1-x^2)^{\frac{-1}{2}}$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left[-\left(1-x^2\right)^{\frac{-1}{2}}\right]$$

$$=\left(-\frac{1}{2}\right)\cdot(1-x^2)^{\frac{-3}{2}}\cdot\frac{d}{dx}(1-x^2)$$

$$=\frac{1}{\sqrt{\left(1-x^2\right)^3}}\times(-2x)$$

$$\Rightarrow\frac{d^2y}{dx^2}=\frac{-x}{\sqrt{(1-x^2)^3}}\quad \text{....(i)}$$

$y=\cos^{-1}x\Rightarrow x=\cos y$

Putting $x=\cos y$ in equation (i), we obtain

$$\frac{d^2y}{dx^2}=\frac{-\cos y}{\sqrt{\left(1-\cos^2 y\right)^3}}$$

$$\Rightarrow\frac{d^2y}{dx^2}=\frac{-\cos y}{\sqrt{\left(\sin^2 y\right)^3}}$$

$$\frac{-\cos y}{\sin^3 y}$$

$$=\frac{-\cos y}{\sin y}\times\frac{1}{\sin^2 y}$$

$$\Rightarrow\frac{d^2y}{dx^2}=\cot y\cdot\operatorname{cosec}^2 y$$

Question 13:

If $y = 3\cos(\log x) + 4\sin(\log x)$, show that $x^2 y_2 + xy_1 + y = 0$

Solution 13:

It is given that, $y = 3\cos(\log x) + 4\sin(\log x)$ and $x^2 y_2 + xy_1 + y = 0$

Then,

$$y_1 = 3 \cdot \frac{d}{dx}[\cos(\log x)] + 4 \cdot \frac{d}{dx}[\sin(\log x)]$$

$$= 3 \cdot \left[-\sin(\log x) \cdot \frac{d}{dx}(\log x)\right] + 4 \cdot \left[\cos(\log x) \cdot \frac{d}{dx}(\log x)\right]$$

$$\therefore y_1 = \frac{-3\sin(\log x)}{x} + \frac{4\cos(\log x)}{x} = \frac{4\cos(\log x) - 3\sin(\log x)}{x}$$

$$\therefore y_2 = \frac{d}{dx}\left(\frac{4\cos(\log x) - 3\sin(\log x)}{x}\right)$$

$$= \frac{x\{4\cos(\log x) - 3\sin(\log x)\}' - \{4\cos(\log x) - 3\sin(\log x)\}(x)'}{x^2}$$

$$= \frac{x\left[4\{\cos(\log x)\}' - \{-3\sin(\log x)\}'\right] - \{4\cos(\log x) - 3\sin(\log x)\} \cdot 1}{x^2}$$

$$= \frac{x\left[-4\sin(\log x) \cdot (\log x)' - 3\cos(\log x)(\log x)'\right] - 4\cos(\log x) + 3\sin(\log x)}{x^2}$$

$$= \frac{x\left[-4\sin(\log x)\frac{1}{x} - 3\cos(\log x)\frac{1}{x}\right] - 4\cos(\log x) + 3\sin(\log x)}{x^2}$$

$$= \frac{-4\sin(\log x) - 3\cos(\log x) - 4\cos(\log x) + 3\sin(\log x)}{x^2}$$

$$= \frac{-\sin(\log x) - 7\cos(\log x)}{x^2}$$

$$\therefore x^2 y_2 + xy_1 + y$$

$$= x^2\left(\frac{-\sin(\log x) - 7\cos(\log x)}{x^2}\right) + x\left(\frac{4\cos(\log x) - 3\sin(\log x)}{x}\right) + 3\cos(\log x) + 4\sin(\log x)$$

$$= -\sin(\log x) - 7\cos(\log x) + 4\cos(\log x) - 3\sin(\log x) + 3\cos(\log x) + 4\sin(\log x)$$

$$= 0$$

Hence, proved.

Question 14:

If $y = Ae^{mx} + Be^{nx}$, show that $\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0$

Solution 14:

It is given that, $y = Ae^{mx} + Be^{nx}$

Then,

$$\frac{dy}{dx} = A \cdot \frac{d}{dx}(e^{mx}) + B \cdot \frac{d}{dx}(e^{nx}) = A \cdot e^{mx} \cdot \frac{d}{dx}(mx) + B \cdot e^{nx} \cdot \frac{d}{dx}(nx) = Ame^{mx} + Bne^{nx}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(Ame^{mx} + Bne^{nx}\right) = Am \cdot \frac{d}{dx}(e^{mx}) + Bn \cdot \frac{d}{dx}(e^{nx})$$

$$= Am \cdot e^{mx} \cdot \frac{d}{dx}(mx) + bn \cdot e^{nx} \cdot \frac{d}{dx}(nx) = Am^2e^{mx} + Bn^2e^{nx}$$

$$\therefore \frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny$$

$$= Am^2e^{mx} + Bn^2e^{nx} - (m+n) \cdot (Ame^{mx} + Bne^{nx}) + mn(Ae^{mx} + Be^{nx})$$

$$= Am^2ex^{mx} + Bn^2e^{nx} - Am^2ex^{mx} - Bmne^{nx} - Amne^{mx} - Bn^2e^{mx} + Amne^{mx} + Bmne^{nx}$$

$$= 0$$

Hence, Proved.

Question 15:

If $y = 500e^{7x} + 600e^{-7x}$, show that $\frac{d^2y}{dx^2} = 49y$

Solution 15:

It is given that, $y = 500e^{7x} + 600e^{-7x}$

Then,

$$\frac{dy}{dx} = 500 \cdot \frac{d}{dx}(e^{7x}) + 600 \cdot \frac{d}{dx}(e^{-7x})$$

$$= 500 \cdot e^{7x} \cdot \frac{d}{dx}(7x) + 600 \cdot e^{-7x} \cdot \frac{d}{dx}(-7x)$$

$$= 3500e^{7x} - 4200e^{-7x}$$

$$\therefore \frac{d^2y}{dx^2} = 3500 \cdot \frac{d}{dx}(e^{7x}) - 4200 \cdot \frac{d}{dx}(e^{-7x})$$

$$= 3500 \cdot e^{7x} \cdot \frac{d}{dx}(7x) - 4200 \cdot e^{-7x} \cdot \frac{d}{dx}(-7x)$$
$$= 7 \times 3500 \cdot e^{7x} + 7 \times 4200 \cdot e^{-7x}$$
$$= 49 \times 500e^{7x} + 49 \times 600e^{-7x}$$
$$= 49\left(500e^{7x} + 600e^{-7x}\right)$$
$$= 49y$$
Hence, proved.

Question 16:

If $e^y(x+1) = 1$, show that $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$

Solution 16:

The given relationship is $e^y(x+1) = 1$

$$e^y(x+1) = 1$$
$$\Rightarrow e^y = \frac{1}{x+1}$$
Taking logarithm on both sides, we obtain
$$y = \log\frac{1}{(x+1)}$$
Differentiating this relationship with respect to x, we obtain
$$\frac{dy}{dx} = (x+1)\frac{d}{dx}\left(\frac{1}{(x+1)}\right) = (x+1) \cdot \frac{-1}{(x+1)^2} = \frac{-1}{x+1}$$
$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx} = \left(\frac{1}{x+1}\right) = -\left(\frac{-1}{(x+1)^2}\right) = \frac{1}{(x+1)^2}$$
$$\Rightarrow \frac{d^2y}{dx^2} = \left(\frac{-1}{x+1}\right)^2$$
$$\Rightarrow \frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$$

Hence, proved.

Question 17:

If $y=(\tan^{-1}x)^2$, show that $\left(x^2+1\right)^2 y_2+2x(x^2+1)y_1=2$

Solution 17:

The given relationship is $y=(\tan^{-1}x)^2$

Then,

$$y_1=2\tan^{-1}x\frac{d}{dx}(\tan^{-1}x)$$

$$\Rightarrow y_1=2\tan^{-1}x\cdot\frac{1}{1+x^2}$$

$$\Rightarrow\left(1+x^2\right)y_1=2\tan^{-1}x$$

Again differentiating with respect to x on both sides, we obtain

$$\left(1+x^2\right)y_2+2xy_1=2\left(\frac{1}{1+x^2}\right)$$

$$\Rightarrow\left(1+x^2\right)y_2+2x(1+x^2)y_1=2$$

Hence, proved.

Exercise 5.8

Question 1:

Verify Rolle's Theorem for the function $f(x)=x^2+2x-8,\ x\in[-4,2]$

Solution 1:

The given function, $f(x)=x^2+2x-8$, being polynomial function, is continuous in $[-4,2]$ and is differentiable in $(-4,2)$.

$$f(-4)=(-4)^2+2x(-4)-8=16-8-8=0$$

$$f(2)=(2)^2+2\times 2-8=4+4-8=0$$

$$\therefore f(-4)=f(2)=0$$

$\Rightarrow$ The value of $f(x)$ at -4 and 2 coincides.

Rolle's Theorem states that there is a point $c\in(-4,2)$ such that $f'(c)=0$

$f(x)=x^2+2x-8$
$\Rightarrow f'(x)=2x+2$
$\therefore f'(c)=0$
$\Rightarrow 2c+2=-1$
$\Rightarrow c=-1$
$c=-1\in(-4,2)$

Hence, Rolle's Theorem is verified for the given function.

Question 2:
Examine if Rolle's Theorem is applicable to any of the following functions. Can you say some thing about the converse of Roller's Theorem from these examples?
i. $f(x)=[x]\ for\ x\in[5,9]$
ii. $f(x)=[x]\ for\ x\in[-2,2]$
iii. $f(x)=x^2-1\ for\ x\in[1,2]$

Solution 2:
By Rolle's Theorem, for a function $f:[a,b]\to R$, if
a) f is continuous on [a, b]
b) f is continuous on (a, b)
c) $f(a)=f(b)$

Then, there exists some $c\in(a,b)$ such that $f'(c)=0$
Therefore, Rolle's Theorem is not applicable to those functions that do not satisfy any of the three conditions of the hypothesis.
(i) $f(x)=[x]\ for\ x\in[5,9]$
It is evident that the given function $f(x)$ is not continuous at every integral point.
In particular, $f(x)$ is not continuous at x = 5 and x = 9
$\Rightarrow f(x)$ is not continuous in [5, 9].
Also $f(5)=[5]=5$ and $f(9)=[9]=9$
$\therefore f(5)\neq f(9)$
The differentiability of f in (5, 9) is checked as follows.
Let n be an integer such that $n\in(5,9)$
The left hand limit limit of *f* at x = n is.
$$\lim_{x\to 0'}\frac{f(n+h)-f(n)}{h}=\lim_{x\to 0'}\frac{[n+h]-[n]}{h}=\lim_{x\to 0'}\frac{n-1-n}{h}=\lim_{x\to 0'}0=0$$
The right hand limit of *f* at x = n is,

$$\lim_{h\to 0'}\frac{f(n+h)-f(n)}{h}=\lim_{h\to 0'}\frac{[n+h]-[n]}{h}=\lim_{h\to 0'}\frac{n-n}{h}=\lim_{h\to 0'}0=0$$

Since the left and right hand limits of *f* at x = n are not equal, *f* is not differentiable at x = n

$\therefore f$ is not differentiable in (5, 9).

It is observed that *f* does not satisfy all the conditions of the hypothesis of Rolle's Theorem.

Hence, Rolle's Theorem is not applicable for $f(x)=[x]\ for\ x\in[5,9]$.

(ii) $f(x)=[x]\ for\ x\in[-2,2]$

It is evident that the given function *f(x)* is not continuous at every integral point.
In particular, *f(x)* is not continuous at $x=-2$ and $x=2$
$\Rightarrow f=(x)$ is not continuous in $[-2,2]$
Also, $f(-2)=[2]=-2$ and $f(2)=[2]=2$
$\therefore f(-2)\neq f(2)$
The differentiability of in $(-2,2)$ is checked as follows.
Let n be an integer such that $n\in(-2,2)$.
The left hand limit of *f* at x = n is,

$$\lim_{h\to 0'}\frac{f(n+h)-f(n)}{h}=\lim_{h\to 0'}\frac{[n+h]-[n]}{h}=\lim_{h\to 0'}\frac{n-1-n}{h}=\lim_{h\to 0'}\frac{-1}{h}=\infty$$

The right hand limit of *f* at x = n is,

$$\lim_{h\to 0'}\frac{f(n+h)-f(n)}{h}=\lim_{h\to 0'}\frac{[n+h]-[n]}{h}=\lim_{h\to 0'}\frac{n-n}{h}=\lim_{h\to 0'}0=0$$

Since the left and right hand limits of *f* at x = n are not equal, *f* is not differentiable at x = n
$\therefore f$ is not continuous in $(-2,2)$.

It is observed that *f* does not satisfy all the conditions of the hypothesis of Rolle's Theorem.
Hence, Roller's Theorem is not applicable for $f(x)=[x]\ for\ x\in[-2,2]$.

(iii) $f(x)=x^2-1\ for\ x\in[1,2]$

It is evident that *f*, being a polynomial function, is continuous in [1, 2] and is differentiable in (1, 2).

$f(1)=(1)^2-1=0$

$f(2)=(2)^2-1=3$

$\therefore f(1)\neq f(2)$

It is observed that *f* does not satisfy a condition of the hypothesis of Roller's Theorem.
Hence, Roller's Theorem is not applicable for $f(x)=x^2-1\ for\ x\in[1,2]$.

Question 3:

If $f:[-5,5]\to R$ is a differentiable function and if $f'(x)$ does not vanish anywhere, then prove that $f(-5)\neq f(5)$.

Solution 3:

It is given that $f:[-5,5]\to R$ is a differentiable function.

Since every differentiable function is a continuous function, we obtain

a) f is continuous on $[-5,5]$.

b) f is continuous on $(-5,5)$.

Therefore, by the Mean Value Theorem, there exists $c\in(-5,5)$ such that

$$f'(c)=\frac{f(5)-f(-5)}{5-(-5)}$$

$$\Rightarrow 10f'(c)=f(5)-f(-5)$$

It is also given that $f'(x)$ does not vanish anywhere.

$$\therefore f'(c)\neq 0$$

$$\Rightarrow 10f'(c)\neq 0$$

$$\Rightarrow f(5)-f(-5)\neq 0$$

$$\Rightarrow f(5)\neq f(-5)$$

Hence, proved.

Question 4:

Verify Mean Value Theorem, if $f(x)=x^2-4x-3$ in the interval [a, b], where a = 1 and b = 4.

Solution 4:

The given function is $f(x)=x^2-4x-3$

f, being a polynomial function, is a continuous in [1, 4] and is differentiable in (1, 4) whose derivative is $2x-4$

$$f(1)=1^2-4\times1-3=6,\ f(4)=4^2-4\times4-3=-3$$

$$\therefore \frac{f(b)-f(a)}{b-a}=\frac{f(4)-f(1)}{4-1}=\frac{-3-(-6)}{3}=\frac{3}{3}=1$$

Mean Value Theorem states that there is a point $c\in(1,4)$ such that $f'(c)=1$

$$f'(c)=1$$

$$\Rightarrow 2c-4=1$$

$$\Rightarrow c=\frac{5}{2},\text{ where } c=\frac{5}{2}\in(1,4)$$

Hence, Mean Value Theorem is verified for the given function.

Question 5:
Verify Mean Value theorem, if $f(x)=x^2-5x^2-3x$ in the interval [a, b], where a = 1 and b = 3. Find all $c \in (1,3)$ for which $f'(c)=0$

Solution 5:
The given function *f* is $f(x)=x^2-5x^2-3x$
f, being a polynomial function, is continuous in [1, 3], and is differentiable in (1, 3)
Whose derivative is $3x^2-10x-3$
$f(1)=1^2-5\times1^2-3\times1=-7,\ f(3)=3^3-3\times3=27$
$$\therefore \frac{f(b)-f(a)}{b-a}=\frac{f(3)-f(1)}{3-1}=\frac{-27-(-7)}{3-1}=-10$$
Mean Value Theorem states that there exist a point $c \in (1,3)$ such that $f'(c)=-10$
$f'(c)=-10$
$\Rightarrow 3c^2-10c-3=10$
$\Rightarrow 3c^2-10c+7=0$
$\Rightarrow 3c^2-3c-7c+7=0$
$\Rightarrow 3c(c-1)-7(c-1)=0$
$\Rightarrow (c-1)(3c-7)=0$
$\Rightarrow c=1, \frac{7}{3}$ where $c=\frac{7}{3}\in(1,3)$
Hence, Mean Value Theorem is verified for the given function and $c=\frac{7}{3}\in(1,3)$ is the only point for which $f'(c)=0$

Question 6:
Examine the applicability of Mean Value Theorem for all three functions given in the above exercise 2.

Solution 6:
Mean Value Theorem states that for a function $f:[a,b]\to R$, if

a) f is continuous on [a, b]
b) f is continuous on (a, b)

Then, there exists some $c \in (a,b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

Therefore, Mean Value Theorem is not applicable to those functions that do not satisfy any of the two conditions of the hypothesis.

(i) $f(x) = [x]$ for $x \in [5,9]$

It is evident that the given function *f (x)* is not continuous at every integral point.

In particular, *f (x)* is not continuous at x = 5 and x = 9

is not continuous in [5, 9].

The differentiability of *f* in (5, 9) is checked as follows,

Let n be an integer such that $n \in (5,9)$.

The left hand limit of *f* at x = n is.

$$\lim_{h\to 0^-} \frac{f(n+h) - f(n)}{h} = \lim_{h\to 0^-} \frac{[n+h]-[n]}{h} = \lim_{h\to 0^-} \frac{n-1-n}{h} = \lim_{h\to 0^-} \frac{-1}{h} = \infty$$

The right hand limit of *f* at x = n is.

$$\lim_{h\to 0^+} \frac{f(n+h) - f(n)}{h} = \lim_{h\to 0^+} \frac{[n+h]-[n]}{h} = \lim_{h\to 0^+} \frac{n-n}{h} = \lim_{h\to 0^+} 0 = 0$$

Since the left and right hand limits of *f* at x = n are not equal, *f* is not differentiable at x = n

$\therefore f$ is not differentiable in (5, 9).

It is observed that *f* does not satisfy all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is not applicable for $f(x) = [x]$ for $x \in [5,9]$

(ii) $f(x) = [x]$ for $x \in [-2,2]$

It is evident that the given function *f (x)* is not continuous at every integral point.

In particular, *f (x)* is not continuous at $x = -2$ and $x = 2$

$\Rightarrow f(x)$ is not continuous in $[-2,2]$.

The differentiability of *f* in $(-2,2)$ is checked as follows.

Let *n* be an integer such that $n \in (-2,2)$.

The left hand limit of *f* at x = n is.

$$\lim_{h\to 0'} \frac{f(n+h) - f(n)}{h} = \lim_{h\to 0'} \frac{[n+h]-[n]}{h} = \lim_{h\to 0'} \frac{n-1-n}{h} = \lim_{h\to 0'} \frac{-1}{h} = \infty$$

The right hand limit of *f* at x = n is.

$$\lim_{h\to 0'} \frac{f(n+h) - f(n)}{h} = \lim_{h\to 0'} \frac{[n+h]-[n]}{h} = \lim_{h\to 0'} \frac{n-n}{h} = \lim_{h\to 0'} 0 = 0$$

Since the left and right hand limits of *f* at x = n are not equal, *f* is not differentiable at x = n

$\therefore f$ is not differentiable in $(-2,2)$.

It is observed that *f* does not satisfy all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is not applicable for $f(x) = [x]$ for $x \in [-2,2]$.

(iii) $f(x)=x^2-1 \; for \; x\in[1,2]$

It is evident that *f*, being a polynomial function, is a continuous in [1, 2] and is differentiable in (1, 2)

It is observed that *f* satisfies all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is applicable for $f(x)=x^2-1 \; for \; x\in[1,2]$

It can be proved as follows.

$f(1)=1^2-1=0, \; f(2)=2^2-1=3$

$$\therefore \frac{f(b)-f(a)}{b-a}=\frac{f(2)-f(1)}{2-1}=\frac{3-0}{1}=3$$

$f'(x)=2x$

$\therefore f'(c)=3$

$\Rightarrow 2c=3$

$\Rightarrow c=\frac{3}{2}=1.5, \text{ where } 1.5\in[1,2]$

Miscellaneous Exercise

Question 1:

Differentiate the function w.r.t x

$\left(3x^2-9x+5\right)^9$

Solution 1:

Let $y=(3x^2-9x+5)^9$

Using chain rule, we obtain

$$\frac{dy}{dx}=\frac{d}{dx}=(3x^2-9x+5)^9$$

$$=9(3x^2-9x+5)^8\cdot\frac{d}{dx}(3x^2-9x+5)$$

$$=9(3x^2-9x+5)^8\cdot(6x-9)$$

$$=9(3x^2-9x+5)^8\cdot 3(2x-3)$$

$$=27(3x^2-9x+5)^8(2x-3)$$

Question 2:
Differentiate the function w.r.t x
$\sin^3 x + \cos^6 x$

Solution 2:
Let $y = \sin^3 x + \cos^6 x$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(\sin^3 x) + \frac{d}{dx}(\cos^6 x)$$

$$= 3\sin^2 x \cdot \frac{d}{dx}(\sin x) + 6\cos^5 x \cdot \frac{d}{dx}(\cos x)$$

$$= 3\sin^2 x \cdot \cos x + 6\cos^5 x \cdot (-\sin x)$$

$$= 3\sin x \cos x(\sin x - 2\cos^4 x)$$

Question 3:
Differentiate the function w.r.t x
$(5x)^{3\cos 2x}$

Solution 3:
Let $y = (5x)^{3\cos 2x}$
Taking logarithm on both sides, we obtain
$\log y = 3\cos 2x \log 5x$
Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx} = 3\left[\log 5x \cdot \frac{d}{dx}(\cos 2x) + \cos 2x \cdot \frac{d}{dx}(\log 5x)\right]$$

$$\Rightarrow \frac{dy}{dx} = 3y\left[\log 5x(-\sin 2x) \cdot \frac{d}{dx}(2x) + \cos 2x \cdot \frac{1}{5x} \cdot \frac{d}{dx}(5x)\right]$$

$$\Rightarrow \frac{dy}{dx} = 3y\left[-2\sin x \log 5x + \frac{\cos 2x}{x}\right]$$

$$\Rightarrow \frac{dy}{dx} = 3y\left[\frac{3\cos 2x}{x} - 6\sin 2x \log 5x\right]$$

$$\therefore \frac{dy}{dx} = (5x)^{3\cos 2x}\left[\frac{3\cos 2x}{x} - 6\sin 2x \log 5x\right]$$

Question 4:

Differentiate the function w.r.t x

$\sin^{-1}\left(x\sqrt{x}\right), 0 \le x \le 1$

Solution 4:

Let $y = \sin^{-1}\left(x\sqrt{x}\right)$

Using chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\sin^{-1}\left(x\sqrt{x}\right)$$

$$= \frac{1}{\sqrt{1-\left(x\sqrt{x}\right)^{3}}} \times \frac{d}{dx}\left(x\sqrt{x}\right)$$

$$= \frac{1}{\sqrt{1-x^{3}}} \cdot \frac{d}{dx}\left(x^{\frac{1}{2}}\right)$$

$$= \frac{1}{\sqrt{1-x^{3}}} \times \frac{3}{2} \cdot x^{\frac{1}{2}}$$

$$= \frac{3\sqrt{x}}{2\sqrt{1-x^{3}}}$$

$$= \frac{3}{2}\sqrt{\frac{x}{1-x^{3}}}$$

Question 5:

Differentiate the function w.r.t x

$\dfrac{\cos^{-1}\frac{x}{2}}{\sqrt{2+7'}}, -2 < x < 2$

Solution 5:

Let $y = \dfrac{\cos^{-1}\frac{x}{2}}{\sqrt{2+7'}}$

By quotient rule, we obtain

$$\frac{dy}{dx}=\frac{\sqrt{2x+7}\frac{d}{dx}\left(\cos^{-1}\frac{x}{2}\right)-\left(\cos^{-1}\frac{x}{2}\right)\frac{d}{dx}\left(\sqrt{2x+7}\right)}{\left(\sqrt{2x+7}\right)^2}$$

$$=\frac{\sqrt{2x+7}\left[\frac{-1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\bullet\frac{d}{dx}\left(\frac{x}{2}\right)\right]-\left(\cos^{-1}\frac{x}{2}\right)\frac{1}{2\sqrt{2x+7}}\bullet\frac{d}{dx}(2x+7)}{2x+7}$$

$$=\frac{\sqrt{2x+7}\frac{-1}{\sqrt{4-x^2}}-\left(\cos^{-1}\frac{x}{2}\right)\frac{2}{2\sqrt{2x+7}}}{2x+7}$$

$$=\frac{-\sqrt{2x+7}}{\sqrt{4-x^2}\,\mathrm{x}(2x+7)}-\frac{\cos^{-1}\frac{x}{2}}{\left(\sqrt{2x+7}\right)(2x+7)}$$

$$=-\left[\frac{1}{\sqrt{4-x^2}\sqrt{2x+7}}+\frac{\cos^{-1}\frac{x}{2}}{(2x+7)^{\frac{3}{2}}}\right]$$

Question 6:

Differentiate the function w.r.t x

$$\cot^{-1}\left[\frac{\sqrt{(1+\sin x)}+\sqrt{(1-\sin x)}}{\sqrt{(1+\sin x)}-\sqrt{(1-\sin x)}}\right],0<x<\frac{\pi}{2}$$

Solution 6:

Let $y=\cot^{-1}\left[\frac{\sqrt{(1+\sin x)}+\sqrt{(1-\sin x)}}{\sqrt{(1+\sin x)}-\sqrt{(1-\sin x)}}\right]$…….(1)

Then, $\left[\frac{\sqrt{(1+\sin x)}+\sqrt{(1-\sin x)}}{\sqrt{(1+\sin x)}-\sqrt{(1-\sin x)}}\right]$

$$=\frac{\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)^2}{\left(\sqrt{1+\sin x}-\sqrt{1-\sin x}\right)\sqrt{1+\sin x}+\sqrt{1-\sin x}}$$

$$=\frac{(1+\sin x)+(1-\sin x)+2\sqrt{(1+\sin x)-(1-\sin x)}}{(1+\sin x)-(1-\sin x)}$$

$$=\frac{2+2\sqrt{1-\sin^2 x}}{2\sin x}$$

$$=\frac{1+\cos x}{\sin x}$$

$$=\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$=\cot\frac{x}{2}$$

Therefore, equation (1) becomes

$$y=\cot^{-1}\left(\cot\frac{x}{2}\right)$$

$$\Rightarrow y=\frac{x}{2}$$

$$\therefore \frac{dy}{dx}=\frac{1}{2}\frac{d}{dx}(x)$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{2}$$

Question 7:
Differentiate the function w.r.t x

$(\log x)^{\log x}, x>1$

Solution 7:
Let $y=(\log x)^{\log x}$

Taking logarithm on both sides, we obtain

$\log y=\log x\cdot\log(\log x)$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\left[\log x \cdot \log(\log x)\right]$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \log(\log x)\cdot\frac{d}{dx}(\log x) + \log x \cdot \frac{d}{dx}[\log(\log x)]$$

$$\Rightarrow \frac{dy}{dx} = y\left[\log(\log x)\cdot\frac{1}{x} + \log x \cdot \frac{1}{\log x}\cdot\frac{d}{dx}(\log x)\right]$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{1}{x}\log(\log x) + \frac{1}{x}\right]$$

$$\therefore \frac{dy}{dx} = (\log x)^{\log x}\left[\frac{1}{x} + \frac{\log(\log x)}{x}\right]$$

Question 8:

Differentiate the function w.r.t x

$\cos(a\cos x + b\sin x)$, for some constant a and b.

Solution 8:

Let $y = \cos(a\cos x + b\sin x)$

By Using chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\cos\left(a\cos x + b\sin x\right)$$

$$\Rightarrow \frac{dy}{dx} = -\sin\left(a\cos x + b\sin x\right)\cdot\frac{d}{dx}\left(a\cos x + b\sin x\right)$$

$$= -\sin\left(a\cos x + b\sin x\right)\cdot\left[a\left(-\sin x\right) + b\cos x\right]$$

$$= \left(a\sin x + b\cos x\right)\cdot\sin\left(a\cos x + b\sin x\right)$$

Question 9:

Differentiate the function w.r.t x

$(\sin x - \cos x)^{(\sin x - \cos x)}, \frac{\pi}{4} < x < \frac{3\pi}{4}$

Solution 9:

Let $y = (\sin x - \cos x)^{(\sin x - \cos x)}$

Taking logarithm on both sides, we obtain

$\log y = \log\left[(\sin x - \cos x)^{(\sin x - \cos x)}\right]$

$\Rightarrow \log y = (\sin x - \cos x)\cdot\log(\sin x - \cos x)$

Differentiating both sides with respect to x, we obtain

$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\left[(\sin x - \cos x)\cdot\log(\sin x - \cos x)\right]$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \log(\sin x - \cos x)\cdot\frac{d}{dx}(\sin x - \cos x) + (\sin x - \cos x)\cdot\frac{d}{dx}\log(\sin x - \cos x)$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \log(\sin x - \cos x)\cdot(\cos x + \sin x) + (\sin x - \cos x)\cdot\frac{1}{(\sin x - \cos x)}\cdot\frac{d}{dx}(\sin x - \cos x)$

$\Rightarrow \frac{dy}{dx} = (\sin x - \cos x)^{(\sin x - \cos x)}\left[(\cos x + \sin x)\cdot\log(\sin x - \cos x) + (\cos x + \sin x)\right]$

$\therefore \frac{dy}{dx} = (\sin x - \cos x)^{(\sin x - \cos x)}(\cos x + \sin x)\left[1 + \log(\sin x - \cos x)\right]$

Question 10:

Differentiate the function w.r.t x

$x^x + x^a + a^x + a^a$, for some fixed a > 0 and x > 0

Solution 10:

Let $y = x^x + x^a + a^x + a^a$

Also, let $x^x = u$, $x^a = v$, $a^x = w$ and $a^a = s$

$\therefore y = u + v + w + s$

$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} + \frac{ds}{dx}$ ....(1)

$u = x^x$

$\Rightarrow \log u = \log x^x$

$\Rightarrow \log u = x\log x$

Differentiating both sides with respect to x, we obtain

$\frac{1}{u}\frac{du}{dx} = \log x\cdot\frac{d}{dx}(x) + x\cdot\frac{d}{dx}(\log x)$

$\Rightarrow \frac{du}{dx} = u\left[\log x\cdot 1 + x\cdot\frac{1}{x}\right]$

$\Rightarrow \frac{du}{dx} = x^x\left[\log x + 1\right] = x^x(1 + \log x)$ ....(2)

$v = x^a$

$\therefore \frac{dv}{dx} = \frac{d}{dx}(x^a)$

$\Rightarrow \frac{dv}{dx} = ax^{a-1}$ …..(3)

$w = a^x$

$\Rightarrow \log w = \log a^x$

$\Rightarrow \log w = x \log a$

Differentiating both sides with respect to x, we obtain

$\frac{1}{w} \cdot \frac{dw}{dx} = \log a \cdot \frac{d}{dx}(x)$

$\Rightarrow \frac{dw}{dx} = w \log a$

$\Rightarrow \frac{dw}{dx} = a^x \log a$ ……(4)

s = a^a

Since a is constant, a^a is also a constant.

$\therefore \frac{ds}{dx} = 0$ …….(5)

From (1), (2), (3), (4), and (5) , we obtain

$$\frac{dy}{dx} = x^x(1+\log x) + ax^{a-1} + a^x \log a + 0$$
$$= x^x\left(1+\log x\right) + ax^{a-1} + a^x \log a$$

Question 11:

Differentiate the function w.r.t x

$x^{x^2-3} + (x-3)^{x^2}$, for $x > 3$

Solution 11:

Let $y = x^{x^2-3} + (x-3)^{x^2}$

Also, let $u = x^{x^2-3}$ and $v = (x-3)^{x^2}$

$\therefore y = u + v$

Differentiating both sides with respect to x, we obtain

$\frac{dv}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ …….(1)

$u = x^{x^2-3}$

$\therefore \log u = \log\left(x^{x^2-3}\right)$

$\log u = \left(x^2-3\right)\log x$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx} = \log x \cdot \frac{d}{dx}\left(x^2-3\right) + \left(x^2-3\right)\cdot\frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \log x \cdot 2x + \left(x^2-3\right)\cdot\frac{1}{3}$$

$$\Rightarrow \frac{du}{dx} = x^{x^2-3}\cdot\left[\frac{x^2-3}{x} + 2\times\log x\right]$$

Also,

$v = (x-3)^{x^2}$

$\therefore \log v = \log(x-3)^{x^2}$

$\Rightarrow \log v = x^2\log(x-3)$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}\cdot\frac{dv}{dx} = \log(x-3)\cdot\frac{d}{dx}\left(x^2\right) + x^2\cdot\frac{d}{dx}[\log(x-3)]$$

$$\Rightarrow \frac{1}{v}\cdot\frac{dv}{dx} = \log(x-3)\cdot 2x + x^2\cdot\frac{1}{x-3}\cdot\frac{d}{dx}(x-3)$$

$$\Rightarrow \frac{dv}{dx} = v\left[2x\log(x-3) + \frac{x^2}{x-3}\cdot 1\right]$$

$$\Rightarrow \frac{dv}{dx} = (x-3)^{x^2}\left[\frac{x^2}{x-3} + 2x\log(x-3)\right]$$

Substituting the expressions of $\frac{du}{dx}$ and $\frac{dv}{dx}$ in equation (1), we obtain

$$\frac{dy}{dx} = x^{x^2-3}\left[\frac{x^2-3}{x} + 2x\log x\right] + (x-3)x^2\left[\frac{x^2}{x-3} + 2x\log(x-3)\right]$$

Question 12:

Find $\frac{dy}{dx}$, if $y = 12(1-\cos t),\ x = 10(t-\sin t),\ \frac{\pi}{2} < t < \frac{\pi}{2}$

$-\frac{\pi}{2} < t < \frac{\pi}{2}$

Solution 12:

It is given that $y=12(1-\cos t),\ x=10(t-\sin t)$

$$\therefore \frac{dx}{dt}=\frac{d}{dt}\left[10(t-\sin t)\right]=10\cdot\frac{d}{dt}(t-\sin t)=10(1-\cos t)$$

$$\frac{dy}{dx}=\frac{d}{dx}[12(1-\cos t)]=12\cdot\frac{d}{dt}(1-\cos t)=12\cdot[0-(-\sin t)]=12\sin t$$

$$\therefore \frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{12\sin t}{10(1-\cos t)}=\frac{12\cdot 2\sin\frac{t}{2}\cdot\cos\frac{t}{2}}{10\cdot 2\sin^2\frac{t}{2}}=\frac{6}{5}\cot\frac{t}{2}$$

Question 13:

Find $\frac{dy}{dx}$, if $y=\sin^{-1}x+\sin^{-1}\sqrt{1-x^2},-1\le x\le 1$

Solution 13:

It is given that $y=\sin^{-1}x+\sin^{-1}\sqrt{1-x^2}$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}\left[\sin^{-1}x+\sin^{-1}\sqrt{1-x^2}\right]$$

$$\Rightarrow \frac{dy}{dx}=\frac{d}{dx}(\sin^{-1}x)+\frac{d}{dx}\left(\sin^{-1}\sqrt{1-x^2}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}+\frac{1}{\sqrt{1\left(\sqrt{1-x^2}\right)}}\cdot\frac{d}{dx}\left(\sqrt{1-x^2}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}+\frac{1}{x}\cdot\frac{1}{2\sqrt{1-x^2}}\cdot\frac{d}{dx}(1-x^2)$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}+\frac{1}{2x\sqrt{1-x^2}}(-2x)$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-x^2}}$$

$$\therefore \frac{dy}{dx}=0$$

Question 14:

If $x\sqrt{1+y}+y\sqrt{1+x}=0$, for $-1<x<1$, prove that $\frac{dy}{dx}=-\frac{1}{(1+x)^2}$

Solution 14:

It is given that,

$$x\sqrt{1+y}+y\sqrt{1+x}=0$$

$$x\sqrt{1+y}=-y\sqrt{1+x}$$

Squaring both sides, we obtain

$$x^2(1+y)=y^2(1+x)$$
$$\Rightarrow x^2+x^2y=y^2+xy^2$$
$$\Rightarrow x^2-y^2=xy^2-x^2y$$
$$\Rightarrow x^2-y^2=xy(y-x)$$
$$\Rightarrow (x+y)(x-y)=xy(y-x)$$
$$\therefore x+y=-xy$$
$$\Rightarrow (1+x)y=-x$$
$$\Rightarrow y=\frac{-x}{(1+x)}$$

Differentiating both sides with respect to x, we obtain

$$y=\frac{-x}{(1+x)}$$

$$\frac{dy}{dx}=-\frac{(1+x)\frac{d}{dx}(x)-x\frac{d}{dx}(1+x)}{(1+x)^2}=-\frac{(1+x)-x}{(1+x)^2}=-\frac{1}{(1+x)^2}$$

Hence , proved.

Question 15:

If $(x-a)^2+(y-b)^2=c^2$, for some c > 0, prove that $\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$ is a constant independent of a and b

Solution 15:

It is given that, $(x-a)^2+(y-b)^2=c^2$

Differentiating both sides with respect to x, we obtain

$$\frac{d}{dx}\left[(x-a)^2\right]+\frac{d}{dx}\left[(y-b)^2\right]=\frac{d}{dx}(c^2)$$

$$\Rightarrow 2(x-a)\cdot\frac{d}{dx}(x-a)+2(y-b)\cdot\frac{d}{dx}(y-b)=0$$

$$\Rightarrow 2(x-a)\cdot 1+2(y-b)\cdot\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{-(x-a)}{y-b} \qquad ......(1)$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}\left[\frac{-(x-a)}{y-b}\right]$$

$$=-\frac{\left[(y-b)\cdot\frac{d}{dx}(x-a)-(x-a)\cdot\frac{d}{dx}(y-b)\right]}{(y-b)^2}$$

$$=-\left[\frac{(y-b)-(x-a)\cdot\frac{dy}{dx}}{(y-b)^2}\right]$$

$$=-\left[\frac{(y-b)-(x-a)\cdot\left\{\frac{-(x-a)}{y-b}\right\}}{(y-b)^2}\right] \qquad \text{[using (1)]}$$

$$=-\left[\frac{\left(y-b\right)^2+(x+a)^2}{(y-b)^2}\right]$$

$$\therefore \left[\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}\right]^{\frac{3}{2}}=\frac{\left[\left(1+\frac{(x-a)^2}{(y-b)^2}\right)\right]^{\frac{3}{2}}}{-\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^3}\right]}=\frac{\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^3}\right]}$$

$$=-\frac{\left[\frac{c^2}{(y-b)^2}\right]^{\frac{3}{2}}}{\frac{c^2}{(y-b)^3}}=\frac{\frac{c^2}{(y-b)^3}}{-\frac{c^2}{(y-b)^3}}$$

=-c, which is constant and is independent of a and b

Hence, proved.

Question 16:

If $\cos y = x\cos(a+y)$ with $\cos a \neq \pm 1$, prove that $\dfrac{dy}{dx} = \dfrac{\cos^2(a+y)}{\sin a}$

Solution 16:

It is given that, $\cos y = x\cos(a+y)$

$$\therefore \frac{d}{dx} = [\cos y] = \frac{d}{dx}[x\cos(a+y)]$$

$$\Rightarrow -\sin y\frac{dy}{dx} = \cos(a+y)\cdot\frac{d}{dx}(x) + x\cdot\frac{d}{dx}[\cos(a+y)]$$

$$\Rightarrow -\sin y = \frac{dy}{dx} = \cos(a+y) + x\cdot[-\sin(a+y)]\frac{dy}{dx}$$

$$\Rightarrow [x\sin(a+y) - \sin y]\frac{dy}{dx} = \cos(a+y) \qquad \ldots\ldots..(1)$$

Since $\cos y = x\cos(a+y)$, $x = \dfrac{\cos y}{\cos(a+y)}$

Then, equation (1) reduces to

$$\left[\frac{\cos y}{\cos(a+y)}\cdot\sin(a+y) - \sin y\right]\frac{dy}{dx} = \cos(a+y)$$

$$\Rightarrow \left[\cos y.\sin(a+y) - \sin y.\cos(a+y)\right].\frac{dy}{dx} = \cos^2(a+y)$$

$$\Rightarrow \sin(a+y-y)\frac{dy}{dx} = \cos^2(a+b)$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos^2(a+b)}{\sin a}$$

Hence, proved.

Question 17:

If $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$, find $\dfrac{d^2y}{dx^2}$

Solution 17:

It is given that, $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$

$$\therefore \frac{dx}{dt} = a \cdot \frac{d}{dt}(\cos t + t\sin t)$$

$$= a\left[-\sin t + \sin t \cdot \frac{d}{dt}(t) + t \cdot \frac{d}{dt}(\sin t)\right]$$

$$= a\left[-\sin t + \sin t + t\cos t\right] = at\cos t$$

$$\frac{dy}{dt} = a \cdot \frac{d}{dt}(\sin t - t\cos t)$$

$$= a\left[\cos t - \left\{\cos t \cdot \frac{d}{dt}(t) + t \cdot \frac{d}{dt}(\cos t)\right\}\right]$$

$$= a\left[\cos t - \left\{\cos t - t\sin t\right\}\right] = at\sin t$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{at\sin t}{at\cos t} = \tan t$$

Then, $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\tan t) = \sec^2 t \cdot \frac{dt}{dx}$

$$= \sec^2 t \cdot \frac{1}{at\cos t} \qquad \left[\frac{dx}{dt} = at\cos t \Rightarrow \frac{dt}{dx} = \frac{1}{at\cos t}\right]$$

$$= \frac{\sec^3 t}{at}, 0 < t < \frac{\pi}{2}$$

Question 18:

If $f(x) = |x|^3$, show that $f''(x)$ exists for all real x, and find it.

Solution 18:

It is known that, $|x| = \begin{cases} x, \text{ if } x \geq 0 \\ -x, \text{ if } x < 0 \end{cases}$

Therefore, when $x \geq 0$, $f(x) = |x|^3 = x^3$

In this case, $f'(x) = 3x^2$ and hence, $f''(x) = 6x$

When $x < 0$, $f(x) = |x|^3 = (-x^3) = x^3$

in this case, $f'(x) = 3x^2$ and hence, $f''(x) = 6x$

Thus, for $f(x) = |x|^3$, $f''^{(x)}$ exists for all real x and is given by,

$$f''(x)=\begin{cases}6x, if\ x\ge 0\\ -6x, if\ x<0\end{cases}$$

Question 19:

Using mathematical induction prove that $\frac{d}{dx}(x^n)=nx^{x-1}$ for all positive integers n.

Solution 19:

To prove: $P(n):\frac{d}{dx}(x^n)=nx^{x-1}$ for all positive integers n.

For n = 1,

$P(1):\frac{d}{dx}(x)=1=1\cdot x^{1-1}$

$\therefore\ p(n)$ is true for $n=1$

Let $p(k)$ is true for some positive integer k.

That is , $p(k):\frac{d}{dx}(x^k)=kx^{k-1}$

It is to be proved that p (k + 1) is also true.

Consider $\frac{d}{dx}\left(x^{k+1}\right)=\frac{d}{dx}(x\cdot x^k)$

$x^k\cdot\frac{d}{dx}(x)+x\cdot\frac{d}{dx}(x^k)$

$=x^k\cdot 1+x\cdot k\cdot x^{k-1}$

$=x^k+kx^k$

$=(k+1)\cdot x^k$

$=(k+1)\cdot x^{(k+1)-1}$

Thus, P(k + 1) is true whenever P(k) is true.

Therefore, by the principal of mathematical induction, the statement P(n) is true for every positive integer n.

Hence, proved.

Question 20:

Using the fact that $\sin(A+B)=\sin A\cos B+\cos A\sin B$ ant the differentiation, obtain the sum formula for cosines.

Solution 20:

$\sin(A+B)=\sin A\cos B+\cos A\sin B$

Differentiating both sides with respect to x, we obtain

$$\frac{d}{dx}\left[\sin(A+B)\right]=\frac{d}{dx}(\sin A\cos B)+\frac{d}{dx}(\cos A\sin B)$$

$$\Rightarrow \cos(A+B)\cdot\frac{d}{dx}(A+B)=\cos B\cdot\frac{d}{dx}(\sin A)+\sin A\cdot\frac{d}{dx}(\cos B)$$

$$+\sin B\cdot\frac{d}{dx}(\cos A)+\cos A\cdot\frac{d}{dx}(\sin B)$$

$$\Rightarrow \cos(A+B)\cdot\frac{d}{dx}(A+B)=\cos B\cdot\cos A\frac{d}{dx}+\sin A(-\sin B)\frac{dB}{dx}$$

$$+\sin B(-\sin A)\cdot\frac{dA}{dx}+\cos A\cos B\frac{dB}{dx}$$

$$\Rightarrow \cos(A+B)\left[\frac{dA}{dx}+\frac{dB}{dx}\right]=(\cos A\cos B-\sin A\sin B)\cdot\left[\frac{dA}{dx}+\frac{dB}{dx}\right]$$

$$\therefore \cos(A+B)=\cos A\cos B-\sin A\sin B$$

Question 21:

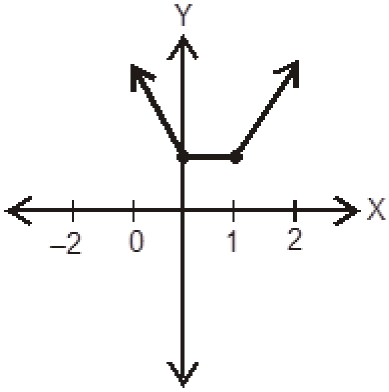
Does there exist a function which is continuous everywhere but not differentiable at exactly two points? Justify your answer.

Solution 21:

Consider $f(x)=|x|+|x+1|$

Since modulus function is everywhere continuous and sum of two continuous function is also continuous.

Differentiability of $f(x)$: Graph of $f(x)$ shows that $f(x)$ is everywhere derivable except possible at $x=0$ and $x=1$



At x = 0, Left hand derivative =

$$\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^-}\frac{(|x|+|x-1|)-(1)}{x}=\lim_{x\to 0^-}\frac{(-x)-(x-1)-1}{x}=\lim_{x\to 0^-}\frac{-2x}{x}=-2$$

Right hand derivative =

$$\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^+}\frac{(|x|+|x-1|)-(1)}{x}=\lim_{x\to 0^+}\frac{(-x)-(x-1)-1}{x}=\lim_{x\to 0^-}\frac{0}{x}=0$$

Since $L.H.D \neq R.H.D$ *f(x)* is not derivable at x = 0.

At x = 1

L.H.D :

$$\lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1^-}\frac{(|x|+|x-1|)}{x-1}=\lim_{x\to 1^-}\frac{(x)-(x-1)-1}{x-1}=\lim_{x\to 1^-}\frac{0}{x-1}=0$$

R.H.D :

$$\lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1^+}\frac{(|x|+|x-1|-1)}{x-1}=\lim_{x\to 1^+}\frac{(x)+(x-1)-1}{x-1}=\lim_{x\to 1^+}\frac{2(x-1)}{x-1}=2$$

Since $L.H.D \neq R.H.D$ *f(x)* is not derivable at x = 1.

$\therefore f(x)$ is continuous everywhere but not derivable at exactly two points.

Question 22:

If $y=\begin{bmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{bmatrix}$, prove that $\dfrac{dy}{dx}=\begin{bmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{bmatrix}$

Solution 22:

$$y=\begin{bmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{bmatrix}$$

$$\Rightarrow y=(mc-nb)f(x)-(lc-na)g(x)+(lb-ma)h(x)$$

Then, $\frac{dy}{dx}=\frac{d}{dx}\left[(mc-nb)f(x)\right]-\frac{d}{dx}\left[(lc-na)g(x)\right]+\frac{d}{dx}\left[(lb-ma)h(x)\right]$

$$=(mc-nb)f'(x)-(lc-na)g'(x)+(lb-ma)h'(x)$$

$$=\begin{bmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{bmatrix}$$

Thus, $\frac{dy}{dx}=\begin{bmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{bmatrix}$

Question 23:

If $y=e^{a\cos^{-1}x}, -1\le x\le 1,$ show that $(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-a^2y=0$

Solution 23:

It is given that, $y=e^{a\cos^{-1}x}$

Taking logarithm on both sides, we obtain

$$\log y = a\cos^{-1}x\log e$$

$$\log y = a\cos^{-1}x$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx}=ax\frac{1}{\sqrt{1-x^2}}$$

$$=\frac{dy}{dx}=\frac{-ay}{\sqrt{1-x^2}}$$

By squaring both the sides, we obtain

$$\left(\frac{dy}{dx}\right)^2 = \frac{a^2y^2}{1-x^2}$$

$$\Rightarrow \left(1-x^2\right)\left(\frac{dy}{dx}\right)^2 = a^2y^2$$

$$\left(1-x^2\right)\left(\frac{dy}{dx}\right)^2 = a^2y^2$$

Again, differentiating both sides with respect to x, we obtain

$$\left(\frac{dy}{dx}\right)^2 \frac{d}{dx}(1-x^2) + (1-x^2)\times\frac{d}{dx}\left[\left(\frac{dy}{dx}\right)^2\right] = a^2\frac{d}{dx}\left(y^2\right)$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2(-2x) + (1-x^2)\times 2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} = a^2\cdot 2y\cdot\frac{dy}{dx}$$

$$\Rightarrow x\frac{dy}{dx} + (1-x^2)\frac{d^2y}{dx^2} = a^2\cdot y \qquad \left[\frac{dy}{dx}\neq 0\right]$$

$$\Rightarrow \left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$$

Hence, proved.