

Ex 29.1

Q1

$$\lim_{x \rightarrow 0} \frac{x}{|x|}$$

We know that $|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$

$$\therefore \lim_{x \rightarrow 0^+} \frac{x}{|x|} = \lim_{x \rightarrow 0^+} \frac{x}{x} = \lim_{x \rightarrow 0^+} 1 = 1$$

$$\text{Also, } \lim_{x \rightarrow 0^-} \frac{x}{|x|} = \lim_{x \rightarrow 0^-} \frac{x}{-x} = \lim_{x \rightarrow 0^-} -1 = -1$$

$$\Rightarrow \text{LHL of } f(x) \neq \text{RHL of } f(x)$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{x}{|x|} \text{ does not exist}$$

Q2

$$\begin{aligned} \lim_{x \rightarrow 2^-} f(x) &= \lim_{x \rightarrow 2^-} (2x + 3) \\ &= 2(2) + 3 \\ &= 7 \end{aligned}$$

$$\therefore \lim_{x \rightarrow 2^-} f(x) = 7$$

Also,

$$\begin{aligned} \lim_{x \rightarrow 2^+} f(x) &= \lim_{x \rightarrow 2^+} (x + k) \\ &= (2 + k) \end{aligned}$$

Since, $\lim_{x \rightarrow 2} f(x)$ exists (given)

$$\therefore \lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x)$$

$$\Rightarrow 7 = 2 + k$$

$$\Rightarrow k = 5$$

Q3

Let $f(x) = \frac{1}{x}$, this function is defined for every value of x except at $x = 0$

$$\text{As } x \rightarrow 0^+, \frac{1}{x} \rightarrow \infty,$$

$$\text{As } x \rightarrow 0^-, \frac{1}{x} \rightarrow -\infty$$

$\therefore \lim_{x \rightarrow 0} \frac{1}{x}$ does not exist.

Q4

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \frac{3x}{-x + 2x} = \lim_{x \rightarrow 0^-} \frac{3x}{x} = 3$$

$$\left[\because \text{as } x \rightarrow 0^-, |x| = -x \right]$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{3x}{x + 2x} = 1$$

$$\left[\because \text{as } x \rightarrow 0^+, |x| = x \right]$$

$$\text{thus, } \lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} f(x)$$

$\therefore \lim_{x \rightarrow 0} f(x)$ does not exist.

Q5

$$\begin{aligned} \lim_{x \rightarrow 0^+} f(x) &= \lim_{x \rightarrow 0^+} x + 1 \\ &= \lim_{h \rightarrow 0} (0 + h) + 1 = 1 \end{aligned}$$

$$\begin{aligned} \text{Also, } \lim_{x \rightarrow 0^-} f(x) &= \lim_{x \rightarrow 0^-} x - 1 \\ &= \lim_{h \rightarrow 0} (0 - h) - 1 = -1 \end{aligned}$$

$$\Rightarrow \lim_{x \rightarrow 0^+} f(x) \neq \lim_{x \rightarrow 0^-} f(x)$$

Hence, limit does not exist.

Q6

$$\begin{aligned}\text{LHL} &= \lim_{x \rightarrow 0^-} f(x) \\ &= \lim_{h \rightarrow 0} f(0-h) \\ &= \lim_{h \rightarrow 0} -h - 4 \\ &= 0 - 4 \\ &= -4\end{aligned}$$

$$\begin{aligned}\text{RHL} &= \lim_{x \rightarrow 0^+} f(x) \\ &= \lim_{h \rightarrow 0} f(0+h) \\ &= \lim_{h \rightarrow 0} h + 5 \\ &= 0 + 5 \\ &= 5\end{aligned}$$

$$\therefore \lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} f(x)$$

Hence $\lim_{x \rightarrow 0} f(x)$ does not exist.

Q7

$$\lim_{x \rightarrow 3^+} f(x) = 4$$

$$\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} (x+1) = \lim_{h \rightarrow 0} (3-h+1) = 3+1 = 4$$

$$\text{Since, } \lim_{x \rightarrow 3^+} f(x) = 4 = \lim_{x \rightarrow 3^-} f(x)$$

$$\therefore \lim_{x \rightarrow 3} f(x) \text{ is } 4$$

Q8

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} 3(x+1) = 3$$

and

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} 2(x) + 3 = 3$$

$$\therefore \lim_{x \rightarrow 0} f(x) = 3$$

$$\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} 3(x+1) = 3+3 = 6$$

$$\therefore \lim_{x \rightarrow 1} f(x) = 6$$

Q9

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} x^2 - 1 = \lim_{h \rightarrow 0} (-1-h)^2 - 1 = 1 - 1 = 0$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} -x^2 - 1 = \lim_{h \rightarrow 0} -(1+h)^2 - 1 = -2$$

Since, $\lim_{x \rightarrow 1^-} f(x) \neq \lim_{x \rightarrow 1^+} f(x)$

$\therefore \lim_{x \rightarrow 1} f(x)$ does not exist.

Q10

$$\begin{aligned}\text{LHL} &= \lim_{x \rightarrow 0^-} f(x) \\ &\Rightarrow \lim_{x \rightarrow 0^-} \frac{|x|}{x} \\ &\Rightarrow \lim_{h \rightarrow 0} \frac{|0-h|}{0-h} \\ &\Rightarrow \lim_{h \rightarrow 0} \frac{+h}{-h} = -1 \quad \text{---(i)}\end{aligned}$$

and,

$$\begin{aligned}\text{RHL} &= \lim_{x \rightarrow 0^+} f(x) \\ &\Rightarrow \lim_{x \rightarrow 0^+} \frac{|x|}{x} \\ &\Rightarrow \lim_{h \rightarrow 0} \frac{|0+h|}{0+h} = \lim_{h \rightarrow 0} \frac{h}{h} = 1 \quad \text{---(ii)}\end{aligned}$$

So, $\text{LHL} \neq \text{RHL}$

$\therefore \lim_{x \rightarrow 0} f(x)$ does not exist.

Q11

$$\begin{aligned}&\lim_{x \rightarrow a_1} f(x) \\ &\Rightarrow \lim_{x \rightarrow a_1} (x - a_1)(x - a_2) \dots (x - a_n) \quad [\text{Putting limit } x \rightarrow a_1] \\ &\Rightarrow (a_1 - a_1)(a_1 - a_2) \dots (a_1 - a_n) \\ &\Rightarrow 0\end{aligned}$$

$$\begin{aligned}\text{And, } &\lim_{x \rightarrow a} f(x) \\ &\Rightarrow \lim_{x \rightarrow a} (x - a_1)(x - a_2) \dots (x - a_n) \quad [\text{Putting limit } x \rightarrow a] \\ &\Rightarrow (a - a_1)(a - a_2) \dots (a - a_n).\end{aligned}$$

Q12

$$\lim_{x \rightarrow 1^+} \frac{1}{(x-1)} = \lim_{h \rightarrow 0} \frac{1}{(1+h-1)} = \lim_{h \rightarrow 0} \frac{1}{h} = \infty$$

Q13(i)

$$\begin{aligned}
\lim_{x \rightarrow 2^+} \frac{x-3}{x^2-4} &= \lim_{h \rightarrow 0} \frac{(2+h)-3}{(2+h)^2-2^2} \\
&= \lim_{h \rightarrow 0} \frac{(2-3+h)}{(2+h-2)(2+h+2)} \\
&= \lim_{h \rightarrow 0} \frac{(h-1)}{(h)(4+h)} \\
&= \lim_{h \rightarrow 0} \frac{1-\frac{1}{h}}{4+h} \\
&= \frac{1-\frac{1}{0}}{4} = -\infty
\end{aligned}$$

$$\left[\because \lim_{x \rightarrow 2^+} f(x) = \lim_{h \rightarrow 0} f(2+h) \right]$$

Q13(ii)

$$\begin{aligned}
\lim_{x \rightarrow 2^-} \frac{x-3}{x^2-4} &= \lim_{h \rightarrow 0} \frac{(2-h)-3}{(2-h)^2-4} \\
&= \lim_{h \rightarrow 0} \frac{(2-h-3)}{(2-h+2)(2-h-2)} \\
&= \lim_{h \rightarrow 0} \frac{-1-h}{(4-h)(-h)} \\
&= \lim_{h \rightarrow 0} \frac{\frac{1}{h}+1}{(4-h)} \\
&= \frac{\frac{1}{0}+1}{4} = \infty
\end{aligned}$$

$$\left[\because \lim_{x \rightarrow 2^-} f(x) = \lim_{h \rightarrow 0} f(2-h) \right]$$

Q13(iii)

$$\begin{aligned}
\lim_{x \rightarrow 0^+} \frac{1}{3x} &= \lim_{h \rightarrow 0} \frac{1}{3(0+h)} \\
&= \lim_{h \rightarrow 0} \frac{1}{0+3h} \\
&= \frac{1}{0} = \infty
\end{aligned}$$

Q13(iv)

$$\begin{aligned}
\lim_{x \rightarrow -8^+} \frac{2x}{x+8} &= \lim_{h \rightarrow 0} \frac{2(-8+h)}{(-8+h)+8} \\
&= \lim_{h \rightarrow 0} \frac{-16+2h}{h} \\
&= \lim_{h \rightarrow 0} \frac{-16}{h} + 2 \\
\Rightarrow \frac{-16}{0} + 2 &= \infty
\end{aligned}$$

Q13(v)

$$\begin{aligned}
\lim_{x \rightarrow 0^+} \frac{2}{\frac{1}{x^5}} &= \lim_{h \rightarrow 0} \frac{2}{(0+h)^{\frac{1}{5}}} \\
\Rightarrow \frac{2}{0} &= \infty
\end{aligned}$$

Q13(vi)

$$\begin{aligned}
\lim_{x \rightarrow \frac{\pi}{2}^-} \tan x &= \lim_{h \rightarrow 0} \tan\left(\frac{\pi}{2} - h\right) \\
&= \tan\left(\frac{\pi}{2} - 0\right) \\
\Rightarrow \tan \frac{\pi}{2} &= \infty
\end{aligned}$$

Q13(vii)

$$\begin{aligned}
& \lim_{x \rightarrow -\frac{\pi}{2}^+} \sec x \\
&= \lim_{h \rightarrow 0} \sec \left(-\frac{\pi}{2} + h \right) \\
&= \sec \left(-\frac{\pi}{2} + 0 \right) \\
&= \sec \left(-\frac{\pi}{2} \right) \\
&= \frac{1}{\cos \left(-\frac{\pi}{2} \right)} \\
&= \frac{-1}{\left(\cos \frac{\pi}{2} \right)} \\
&= \frac{-1}{0} = -\infty
\end{aligned}$$

Q13(viii)

$$\begin{aligned}
& \lim_{x \rightarrow 0^+} \frac{x^2 - 3x + 2}{x^3 - 2x^2} \\
&= \lim_{x \rightarrow 0^+} \frac{x^2 - x - 2x + 2}{x^2(x - 2)} \\
&= \lim_{x \rightarrow 0^+} \frac{x(x - 1) - 2(x - 1)}{x^2(x - 2)} \\
&= \lim_{x \rightarrow 0^+} \frac{(x - 1)(x - 2)}{x^2(x - 2)} \\
&= \lim_{x \rightarrow 0^+} \frac{(x - 1)}{x^2} \\
&= \lim_{h \rightarrow 0} \frac{(0 - h - 1)}{(0 - h)^2} \\
&= \frac{-h}{h^2} = \frac{-1}{h} = \frac{-1}{0} = -\infty
\end{aligned}$$

Q13(ix)

$$\begin{aligned}
\lim_{x \rightarrow -2^+} \frac{x^2 - 1}{2x + 4} &= \lim_{x \rightarrow -2^+} \frac{(x-1)(x+1)}{2(x+2)} \\
&= \lim_{h \rightarrow 0} \frac{(-2+h-1)(-2+h+1)}{2(-2+h+2)} \\
&= \lim_{h \rightarrow 0} \frac{(-3+h)(h-1)}{2h} \\
\Rightarrow \quad \frac{-3 \times -1}{2 \times 0} &= \frac{1}{0} = \infty
\end{aligned}$$

Q13(x)

$$\begin{aligned}
\lim_{x \rightarrow 0^-} 2 - \cot x &= \lim_{h \rightarrow 0} 2 - \cot(0 - h) \\
&= \lim_{h \rightarrow 0} 2 - (-1) \coth \\
&= \lim_{h \rightarrow 0} 2 + \coth \\
&= \lim_{h \rightarrow 0} 2 + \frac{1}{\tanh} \\
\Rightarrow \quad 2 + \frac{1}{0} &\Leftarrow \infty
\end{aligned}$$

Q13(xi)

$$\begin{aligned}
\lim_{x \rightarrow 0^-} 1 + \operatorname{cosec} x &= \lim_{x \rightarrow 0^-} 1 + \operatorname{cosec}(0 - h) \\
&= \lim_{h \rightarrow 0} 1 - \operatorname{cosec} h \\
&= \lim_{h \rightarrow 0} 1 - \frac{1}{\sinh} \\
\Rightarrow \quad 1 - \frac{1}{0} &= -\infty
\end{aligned}$$

Q14

$$\lim_{x \rightarrow 0} e^{\frac{-1}{x}}$$

$$\lim_{x \rightarrow 0^+} e^{\frac{-1}{x}} = \lim_{x \rightarrow 0^+} \frac{1}{e^{\frac{1}{x}}} = \lim_{h \rightarrow 0} \frac{1}{e^{\frac{1}{0+h}}} = \frac{1}{e^{\frac{1}{0}}} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0$$

$$\text{And, } \lim_{x \rightarrow 0^-} e^{\frac{-1}{x}} = \lim_{x \rightarrow 0^-} \frac{1}{e^{\frac{1}{x}}} = \lim_{h \rightarrow 0} \frac{1}{e^{\frac{1}{0-h}}} = \lim_{h \rightarrow 0} \frac{1}{e^{-\frac{1}{h}}} = \frac{1}{e^{-\frac{1}{0}}} = \frac{1}{e^{-\infty}} = e^{\infty} = \infty$$

$$\Rightarrow \lim_{x \rightarrow 0^+} e^{\frac{-1}{x}} \neq \lim_{x \rightarrow 0^-} e^{\frac{-1}{x}}$$

$$\therefore \lim_{x \rightarrow 0} e^{\frac{-1}{x}} \text{ does not exist.}$$

Q15

$$\begin{aligned} \text{(i)} \quad \lim_{x \rightarrow 2} [x] \\ \lim_{x \rightarrow 2^-} [x] &= 1 \\ \lim_{x \rightarrow 2^+} [x] &= 2 \end{aligned}$$

Thus, $\lim_{x \rightarrow 2} [x]$ does not exist.

$$\begin{aligned} \text{(ii)} \quad \lim_{x \rightarrow \frac{5}{2}} [x] \\ \lim_{x \rightarrow \frac{5}{2}^+} [x] &= 2 \\ \lim_{x \rightarrow \frac{5}{2}^-} [x] &= 2 \end{aligned}$$

$$\Rightarrow \lim_{x \rightarrow \frac{5}{2}} [x] = 2$$

$$\begin{aligned} \text{(iii)} \quad \lim_{x \rightarrow 1} [x] \\ \lim_{x \rightarrow 1^-} [x] &= 0 \\ \lim_{x \rightarrow 1^+} [x] &= 1 \end{aligned}$$

$$\Rightarrow \lim_{x \rightarrow 1^-} [x] \neq \lim_{x \rightarrow 1^+} [x]$$

Thus, $\lim_{x \rightarrow 1} [x]$ does not exist

Q16

$$\begin{aligned} &\lim_{x \rightarrow a^+} [x] \\ \Rightarrow \quad \lim_{h \rightarrow 0^+} [a+h] &= [a] \\ \Rightarrow \quad \lim_{h \rightarrow 0^+} [x] &= [a] \forall a \in \mathbb{R} \\ \text{Also, } \lim_{x \rightarrow 1^-} [x] &= \lim_{h \rightarrow 0^+} [1-h] \\ &= 0 \\ \Rightarrow \quad \lim_{x \rightarrow 1^-} [x] &= 0 \end{aligned}$$

Q17

$$\lim_{x \rightarrow 2^-} \frac{x}{[x]} = \lim_{x \rightarrow 2^-} \frac{x}{1} = \frac{2}{1} = 2$$

$$\left[\because \lim_{x \rightarrow k^-} [x] = k - 1 \right]$$

$$\text{Also, } \lim_{x \rightarrow 2^+} \frac{x}{[x]} = \lim_{x \rightarrow 2^+} \frac{x}{3} = \frac{2}{3}$$

$$\left[\lim_{x \rightarrow k^+} [x] = k + 1 \right]$$

$$\Rightarrow \lim_{x \rightarrow 2^-} \frac{x}{[x]} \neq \lim_{x \rightarrow 2^+} \frac{x}{[x]}$$

Q18

$$\lim_{x \rightarrow 3^+} \frac{x}{[x]} = \lim_{x \rightarrow 3^+} \frac{x}{3} = \frac{3}{3} = 1$$

$$\lim_{x \rightarrow 3^-} \frac{x}{[x]} = \lim_{x \rightarrow 3^-} \frac{x}{2} = \frac{3}{2} = 1.5$$

$$\text{Therefore, } \lim_{x \rightarrow 3^+} \frac{x}{[x]} \neq \lim_{x \rightarrow 3^-} \frac{x}{[x]}$$

Q19

$$\lim_{x \rightarrow \frac{5}{2}} [x]$$

$$\lim_{x \rightarrow \frac{5}{2}} [x] = \left[\frac{5}{2} \right],$$

$$= [2.5] = 2$$

[By definition of greatest integer function]

$$\Rightarrow \lim_{x \rightarrow \frac{5}{2}} [x] = 2$$

Q20

$$\begin{aligned}\lim_{x \rightarrow 2^-} f(x) &= \lim_{x \rightarrow 2^-} (x - [x]) \\ &= \lim_{x \rightarrow 2^-} x - \lim_{x \rightarrow 2^-} [x] \\ &= 2 - 1 = 1\end{aligned}$$

$$\left[\because \lim_{x \rightarrow k^-} [x] = k - 1 \right]$$

$$\begin{aligned}\lim_{x \rightarrow 2^+} f(x) &= \lim_{x \rightarrow 2^+} (3x - 5) \\ &= 3(2) - 5 \\ &= 6 - 5 \\ &= 1\end{aligned}$$

$$\left[\because x > 2 \right]$$

$$\text{Thus, } \lim_{x \rightarrow 2^-} f(x) = 1 = \lim_{x \rightarrow 2^+} f(x)$$

$$\Rightarrow \lim_{x \rightarrow 2} f(x) = 1$$

Q21

$$\begin{aligned}\lim_{x \rightarrow 0^-} \sin \frac{1}{x} &= \lim_{h \rightarrow 0} \sin \frac{1}{0-h} = - \lim_{h \rightarrow 0} \sin \frac{1}{h} \\ &= - (\text{An oscillating number which oscillates between } -1 \text{ and } 1),\end{aligned}$$

So, $\lim_{x \rightarrow 0^-} \sin \frac{1}{x}$ does not exist.

Similarly, $\lim_{x \rightarrow 0^+} \sin \frac{1}{x}$ does not exist.

$\lim_{x \rightarrow 0} \sin \frac{1}{x}$ does not exist.

Q22

Let $f(x) = \begin{cases} \frac{k \cos x}{\pi - 2x}, & \text{where } x \neq \frac{\pi}{2} \\ 3 & , \text{ where } x = \frac{\pi}{2} \end{cases}$ and if $\lim_{x \rightarrow \frac{\pi}{2}} f(x) = f\left(\frac{\pi}{2}\right)$, find the value of k .

$$\text{LHL} = \lim_{x \rightarrow \frac{\pi}{2}} f(x)$$

$$= \lim_{h \rightarrow \frac{\pi}{2}} f\left(\frac{\pi}{2} - h\right)$$

$$= \lim_{h \rightarrow \frac{\pi}{2}} \frac{k \cos\left(\frac{\pi}{2} - h\right)}{\pi - 2\left(\frac{\pi}{2} - h\right)}$$

$$= \lim_{h \rightarrow \frac{\pi}{2}} \frac{k \cos\left(\frac{\pi}{2} - h\right)}{2h}$$

$$= \frac{k \cos\left(\frac{\pi}{2} - \frac{\pi}{2}\right)}{\pi}$$

$$= \frac{k}{\pi}$$

Ex 29.2

Q1

$$\lim_{x \rightarrow 1} \frac{x^2 + 1}{x + 1} = \frac{(1)^2 + 1}{1 + 1} = \frac{1 + 1}{2} = \frac{2}{2} = 1$$

Q2

$$\lim_{x \rightarrow 0} \frac{2x^2 + 3x + 4}{x^2 + 3x + 2} = \lim_{x \rightarrow 0} \frac{2x^2 + 3x + 4}{(x + 2)(x + 1)} = \frac{2(0) + 3(0) + 4}{(0 + 2)(0 + 1)} = \frac{4}{2} = 2$$

Q3

$$\lim_{x \rightarrow 3} \frac{\sqrt{2x + 3}}{x + 3} = \frac{\sqrt{9}}{6} = \frac{1}{2}$$

Q4

$$\lim_{x \rightarrow 1} \frac{\sqrt{x + 8}}{\sqrt{x}} = \frac{\sqrt{(1 + 8)}}{\sqrt{1}} = \sqrt{9} = 3$$

Q5

$$\lim_{x \rightarrow a} \frac{\sqrt{x} + \sqrt{a}}{x + a} = \frac{\sqrt{a} + \sqrt{a}}{a + a} = \frac{2\sqrt{a}}{2a} = \frac{1}{\sqrt{a}}$$

Q6

$$\lim_{x \rightarrow 1} \frac{1 + (x - 1)^2}{1 + x^2} = \frac{1 + 0^2}{1 + 1} = \frac{1}{2}$$

Q7

$$\lim_{x \rightarrow 0} \frac{x^{\frac{2}{3}} - 9}{x - 27} = \frac{-9}{-27} = \frac{1}{3}$$

Q8

$$\lim_{x \rightarrow 0} 9 = 9$$

Q9

$$\lim_{x \rightarrow 2} (3 - x) = (3 - 2) = 1$$

Q10

$$\lim_{x \rightarrow -1} (4x^2 + 2) = (4(-1)^2 + 2) = 6$$

Q11

$$\lim_{x \rightarrow -1} \frac{x^3 - 3x + 1}{x - 1} = \frac{(-1)^3 - 3(-1) + 1}{(-1) - 1} = \frac{-1 + 3 + 1}{-2} = \frac{3}{-2} = -\frac{3}{2}$$

Q12

$$\lim_{x \rightarrow 0} \frac{3x + 1}{x + 3} = \frac{3(0) + 1}{(0 + 3)} = \frac{1}{3}$$

Q13

$$\lim_{x \rightarrow 3} \frac{x^2 - 9}{x + 2} = \frac{3^2 - 9}{3 + 2} = 0$$

Q14

$$\lim_{x \rightarrow 0} \frac{ax + b}{(x + d)} = \frac{a \times 0 + b}{(0 + d)} = \frac{b}{d}$$

Ex 29.3

Q1

$$\lim_{x \rightarrow -5} \frac{2x^2 + 9x - 5}{x + 5} = \lim_{x \rightarrow -5} \frac{(x + 5)(2x - 1)}{(x + 5)} = \lim_{x \rightarrow -5} (2x - 1) = 2(-5) - 1 = -10 - 1 = -11$$

Q2

$$\begin{aligned} \lim_{x \rightarrow 3} \frac{x^2 - 4x + 3}{x^2 - 2x - 3} &= \lim_{x \rightarrow 3} \frac{x^2 - 3x - x + 3}{x^2 + x - 3x - 3} \\ &= \lim_{x \rightarrow 3} \frac{x(x - 1) - 3(x - 1)}{x(x + 1) - 3(x + 1)} \\ &= \lim_{x \rightarrow 3} \frac{(x - 1)(x - 3)}{(x + 1)(x - 3)} \\ &= \lim_{x \rightarrow 3} \frac{x - 1}{x + 1} \\ &= \frac{3 - 1}{3 + 1} \\ &= \frac{2}{4} = \frac{1}{2} \end{aligned}$$

Q3

$$\lim_{x \rightarrow 3} \frac{x^4 - 81}{x^2 - 9} = \lim_{x \rightarrow 3} \frac{(x^2 - 9)(x^2 + 9)}{(x^2 - 9)} = \lim_{x \rightarrow 3} x^2 + 9 = (3)^2 + 9 = 9 + 9 = 18$$

Q4

$$\lim_{x \rightarrow 2} \frac{x^3 - 8}{x^2 - 4} = \lim_{x \rightarrow 2} \frac{(x - 2)(x^2 + 4 + 2x)}{(x - 2)(x + 2)} = \frac{(2)^2 + 4 + 2(2)}{2 + 2} = \frac{4 + 4 + 4}{4} = \frac{12}{4} = 3$$

Q5

$$\begin{aligned}\lim_{x \rightarrow -\frac{1}{2}} \frac{8x^3 + 1}{2x + 1} &= \lim_{x \rightarrow -\frac{1}{2}} \frac{8 \left(x^3 + \frac{1}{8} \right)}{2 \left(x + \frac{1}{2} \right)} \\&= \frac{8}{2} \lim_{x \rightarrow -\frac{1}{2}} \frac{\left(x^3 + \left(\frac{1}{2} \right)^3 \right)}{x + \frac{1}{2}} \\&= 4 \lim_{x \rightarrow -\frac{1}{2}} \frac{\left(x + \frac{1}{2} \right) \left(x^2 + \frac{1}{4} - \frac{1}{2}x \right)}{\left(x + \frac{1}{2} \right)} \\&= 4 \left(\left(\frac{-1}{2} \right)^2 + \frac{1}{4} - \frac{1}{2} \left(\frac{-1}{2} \right) \right) \\&= 4 \left(\frac{1}{4} + \frac{1}{4} + \frac{1}{4} \right) \\&= 3\end{aligned}$$

Q6

$$\begin{aligned}\lim_{x \rightarrow 4} \frac{x^2 - 7x + 12}{x^2 - 3x - 4} &= \lim_{x \rightarrow 4} \frac{x^2 - 3x - 4x + 12}{x^2 + x - 4x - 4} \\&= \lim_{x \rightarrow 4} \frac{x(x-3) - 4(x-3)}{x(x+1) - 1(x+1)} \\&= \lim_{x \rightarrow 4} \frac{(x-3)(x-4)}{(x-4)(x+1)} \\&= \lim_{x \rightarrow 4} \frac{x-3}{x+1} \\&= \frac{4-3}{4+1} \\&= \frac{1}{5}\end{aligned}$$

Q7

$$\begin{aligned}\lim_{x \rightarrow 2} \frac{x^4 - 16}{x - 2} &= \lim_{x \rightarrow 2} \frac{(x^2 - 4)(x^2 + 4)}{(x - 2)} \\&= \lim_{x \rightarrow 2} \frac{(x - 2)(x + 2)(x^2 + 4)}{(x - 2)} \\&= \lim_{x \rightarrow 2} (x + 2)(x^2 + 4) \\&= (2 + 2)(4 + 4) \\&= 4(8) \\&= 32\end{aligned}$$

Q8

$$\begin{aligned}\lim_{x \rightarrow 5} \frac{x^2 - 9x + 20}{x^2 - 6x + 5} &= \lim_{x \rightarrow 5} \frac{x^2 - 4x - 5x + 20}{x^2 - x - 5x + 5} \\&= \lim_{x \rightarrow 5} \frac{x(x - 4) - 5(x - 4)}{x(x - 1) - 5(x - 1)} \\&= \lim_{x \rightarrow 5} \frac{(x - 5)(x - 4)}{(x - 5)(x - 1)} \\&= \lim_{x \rightarrow 5} \frac{x - 4}{x - 1} \\&= \frac{5 - 4}{5 - 1} \\&= \frac{1}{4}\end{aligned}$$

Q9

$$\begin{aligned}\lim_{x \rightarrow -1} \frac{x^3 + 1}{x + 1} &= \lim_{x \rightarrow -1} \frac{(x + 1)(x^2 - x + 1)}{(x + 1)} && \left[a^3 + b^3 = (a + b)(a^2 + b^2 - ab) \right] \\&= \lim_{x \rightarrow -1} (x^2 - x + 1) \\&= (-1)^2 - (-1) + 1 \\&= 1 + 1 + 1 \\&= 3\end{aligned}$$

Q10

$$\lim_{x \rightarrow 5} \frac{x^3 - 125}{x^2 - 7x + 10} = \lim_{x \rightarrow 5} \frac{(x - 5)(x^2 + 25 + 5x)}{(x - 2)(x - 5)} = \frac{(5)^2 + 25 + 5(5)}{(5 - 2)} = \frac{25 + 25 + 25}{3} = \frac{75}{3} = 25$$

Q11

$$\begin{aligned}\lim_{x \rightarrow \sqrt{2}} \frac{x^2 - 2}{x^2 + \sqrt{2}x - 4} &= \lim_{x \rightarrow \sqrt{2}} \frac{(x - \sqrt{2})(x + \sqrt{2})}{x^2 + 2\sqrt{2}x - \sqrt{2}x - 4} \\&= \lim_{x \rightarrow \sqrt{2}} \frac{(x - \sqrt{2})(x + \sqrt{2})}{x(x + 2\sqrt{2}) - \sqrt{2}(x + 2\sqrt{2})} \\&= \lim_{x \rightarrow \sqrt{2}} \frac{(x - \sqrt{2})(x + \sqrt{2})}{(x + 2\sqrt{2})(x - \sqrt{2})} \\&= \frac{\sqrt{2} + \sqrt{2}}{\sqrt{2} + 2\sqrt{2}} = \frac{2\sqrt{2}}{3\sqrt{2}} \\&= \frac{2}{3}\end{aligned}$$

Q12

$$\begin{aligned}\lim_{x \rightarrow \sqrt{3}} \frac{x^2 - 3}{x^2 + 3\sqrt{3}x - 12} &= \lim_{x \rightarrow \sqrt{3}} \frac{(x - \sqrt{3})(x + \sqrt{3})}{x^2 + 4\sqrt{3}x - \sqrt{3}x - 12} \\&= \lim_{x \rightarrow \sqrt{3}} \frac{(x - \sqrt{3})(x + \sqrt{3})}{x(x + 4\sqrt{3}) - \sqrt{3}(x + 4\sqrt{3})} \\&= \lim_{x \rightarrow \sqrt{3}} \frac{(x - \sqrt{3})(x + \sqrt{3})}{(x - \sqrt{3})(x + 4\sqrt{3})} \\&= \frac{\sqrt{3} + \sqrt{3}}{\sqrt{3} + 4\sqrt{3}} = \frac{2\sqrt{3}}{5\sqrt{3}} \\&= \frac{2}{5}\end{aligned}$$

Q13

$$\begin{aligned}\lim_{x \rightarrow \sqrt{3}} \frac{x^4 - 9}{x^2 + 4\sqrt{3}x - 15} &= \lim_{x \rightarrow \sqrt{3}} \frac{(x - \sqrt{3})(x + \sqrt{3})(x^2 + 3)}{(x - \sqrt{3})(x + 5\sqrt{3})} = \lim_{x \rightarrow \sqrt{3}} \frac{(x + \sqrt{3})(x^2 + 3)}{(x + 5\sqrt{3})} \\&= \frac{(\sqrt{3} + \sqrt{3})(3 + 3)}{(\sqrt{3} + 5\sqrt{3})} = \frac{(2\sqrt{3})(6)}{6\sqrt{3}} = 2\end{aligned}$$

Q14

$$\begin{aligned} & \lim_{x \rightarrow 2} \left(\frac{x}{x-2} - \frac{4}{x^2-2x} \right) \\ &= \lim_{x \rightarrow 2} \left(\frac{x}{x-2} - \frac{4}{x(x-2)} \right) \\ &= \lim_{x \rightarrow 2} \left(\frac{x(x) - 4}{x(x-2)} \right) \\ &= \lim_{x \rightarrow 2} \left(\frac{x^2 - 4}{x(x-2)} \right) \\ &= \lim_{x \rightarrow 2} \frac{(x-2)(x+2)}{x(x-2)} \\ &= \lim_{x \rightarrow 2} \frac{(x+2)}{x} \\ &= \frac{2+2}{2} \\ &= \frac{4}{2} \\ &= 2 \end{aligned}$$

Q15

$$\begin{aligned} & \lim_{x \rightarrow 1} \left(\frac{1}{x^2+x-2} - \frac{x}{x^3-1} \right) \\ &= \lim_{x \rightarrow 1} \left(\frac{1}{x^2+x-2} - \frac{x}{(x-1)(x^2+x+1)} \right) \\ &= \lim_{x \rightarrow 1} \left(\frac{x^3-1-x^3-x^2+2x}{(x^3-1)(x^2+x-2)} \right) \\ &= \lim_{x \rightarrow 1} \left(- \frac{(x^2-2x+1)}{(x^3-1)(x^2+x-2)} \right) \\ &= \lim_{x \rightarrow 1} \left(- \frac{(x-1)(x-1)}{(x-1)(x^2+1+x)(x^2+x-2)} \right) \\ &= - \lim_{x \rightarrow 1} \left(\frac{x-1}{(x^2+1+x)(x+2)(x-1)} \right) \\ &= - \frac{1}{(1+1+1)(1+2)} \\ &= - \frac{1}{9} \end{aligned}$$

Q16

$$\begin{aligned}\lim_{x \rightarrow 3} \left(\frac{1}{x-3} - \frac{2}{x^2-4x+3} \right) \\&= \lim_{x \rightarrow 3} \left(\frac{1}{x-3} - \frac{2}{(x-3)(x-1)} \right) \\&= \lim_{x \rightarrow 3} \left(\frac{x-1-2}{(x-1)(x-3)} \right) \\&= \lim_{x \rightarrow 3} \left(\frac{x-3}{(x-1)(x-3)} \right) \\&= \lim_{x \rightarrow 3} \frac{1}{x-1} \\&= \frac{1}{3-1} \\&= \frac{1}{2}\end{aligned}$$

Q17

$$\begin{aligned}\lim_{x \rightarrow 2} \left(\frac{1}{x-2} - \frac{2}{x^2-2x} \right) \\&= \lim_{x \rightarrow 2} \left(\frac{1}{x-2} - \frac{2}{x(x-2)} \right) \\&= \lim_{x \rightarrow 2} \left(\frac{x-2}{(x-2)(x)} \right) \\&= \lim_{x \rightarrow 2} \left(\frac{\cancel{x-2}}{\cancel{x-2}(x)} \right) \\&= \lim_{x \rightarrow 2} \frac{1}{x} \\&= \frac{1}{2}\end{aligned}$$

Q18

$$\lim_{x \rightarrow \frac{1}{4}} \frac{4x-1}{2\sqrt{x}-1} = \lim_{x \rightarrow \frac{1}{4}} \frac{4\left(x - \frac{1}{4}\right)}{2\left(\sqrt{x} - \frac{1}{2}\right)} = 4 \lim_{x \rightarrow \frac{1}{4}} \frac{\left(\sqrt{x} - \frac{1}{2}\right)\left(\sqrt{x} + \frac{1}{2}\right)}{2\left(\sqrt{x} - \frac{1}{2}\right)} = 4 \lim_{x \rightarrow \frac{1}{4}} \frac{\left(\sqrt{x} + \frac{1}{2}\right)}{2} = \frac{4\left(\frac{1}{2} + \frac{1}{2}\right)}{2} = \frac{4(1)}{2} = 2$$

Q19

$$\begin{aligned}
\lim_{x \rightarrow 4} \frac{x^2 - 16}{\sqrt{x} - 2} &= \lim_{x \rightarrow 4} \frac{(x - 4)(x + 4)}{(\sqrt{x} - 2)} \\
&= \lim_{x \rightarrow 4} \frac{(\sqrt{x} - 2)(\sqrt{x} + 2)(x + 4)}{(\sqrt{x} - 2)} \\
&= \lim_{x \rightarrow 4} (\sqrt{x} + 2)(x + 4) \\
&= (2 + 2)(4 + 4) \\
&= 4(8) \\
&= 32
\end{aligned}$$

Q20

$$\begin{aligned}
\lim_{x \rightarrow 0} \frac{(a + x)^2 - a^2}{x} &= \lim_{x \rightarrow 0} \frac{(a + x - a)(a + x + a)}{x} \\
&= \lim_{x \rightarrow 0} \frac{(x)(2a + x)}{x} \\
&= \lim_{x \rightarrow 0} (2a + x) \\
&= 2a
\end{aligned}$$

Q21

$$\begin{aligned}
\lim_{x \rightarrow 2} \left(\frac{1}{x - 2} - \frac{4}{x^3 - 2x^2} \right) &= \lim_{x \rightarrow 2} \left(\frac{1}{x - 2} - \frac{4}{x^2(x - 2)} \right) \\
&= \lim_{x \rightarrow 2} \left(\frac{x^2 - 4}{x^2(x - 2)} \right) \\
&= \lim_{x \rightarrow 2} \frac{(x - 2)(x + 2)}{x^2(x - 2)} \\
&= \lim_{x \rightarrow 2} \frac{(x + 2)}{x^2} \\
&= \frac{(2 + 2)}{2^2} = \frac{4}{4} \\
&= 1
\end{aligned}$$

Q22

$$\begin{aligned}
& \lim_{x \rightarrow 3} \left(\frac{1}{x-3} - \frac{3}{x^2-3x} \right) \\
&= \lim_{x \rightarrow 3} \left(\frac{1}{x-3} - \frac{3}{x(x-3)} \right) \\
&= \lim_{x \rightarrow 3} \left(\frac{x-3}{x(x-3)} \right) \\
&= \lim_{x \rightarrow 3} \left(\frac{1}{x} \right) \\
&= \frac{1}{3}
\end{aligned}$$

Q23

$$\begin{aligned}
& \lim_{x \rightarrow 1} \left(\frac{1}{x-1} - \frac{2}{x^2-1} \right) \\
&= \lim_{x \rightarrow 1} \left(\frac{1}{x-1} - \frac{2}{(x-1)(x+1)} \right) \\
&= \lim_{x \rightarrow 1} \left(\frac{x+1-2}{(x-1)(x+1)} \right) \\
&= \lim_{x \rightarrow 1} \left(\frac{x-1}{(x-1)(x+1)} \right) \\
&= \lim_{x \rightarrow 1} \left(\frac{1}{x+1} \right) \\
&= \frac{1}{1+1} = \frac{1}{2}
\end{aligned}$$

Q24

$$\begin{aligned}
& \lim_{x \rightarrow 3} (x^2 - 9) \left[\frac{1}{x+3} + \frac{1}{x-3} \right] \\
&= \lim_{x \rightarrow 3} (x^2 - 9) \left[\frac{x-3+x+3}{(x+3)(x-3)} \right] \\
&= \lim_{x \rightarrow 3} (x^2 - 9) \left[\frac{2x}{(x+3)(x-3)} \right] \\
&= \lim_{x \rightarrow 3} \frac{(x-3)(x+3)(2x)}{(x+3)(x-3)} \\
&= \lim_{x \rightarrow 3} 2(x) = 2(3) = 6
\end{aligned}$$

Q25

$$\lim_{x \rightarrow 1} \frac{x^4 - 3x^3 + 2}{x^3 - 5x^2 + 3x + 1}$$

Dividing $x^4 - 3x^3 + 2$ by $x^3 - 5x^2 + 3x + 1$

$$\begin{array}{r} x^3 - 5x^2 + 3x + 1 \overline{) x^4 - 3x^3 + 2} \\ \underline{+x^4 - 5x^3 + 3x^2 + x} \\ 2x^3 - 3x^2 - x + 2 \\ \underline{+2x^3 - 10x^2 + 6x + 2} \\ 7x^2 - 7x \end{array}$$

$$\begin{aligned} \Rightarrow \lim_{x \rightarrow 1} \frac{x^4 - 3x^3 + 2}{x^3 - 5x^2 + 3x + 1} &= \lim_{x \rightarrow 1} x + 2 + \lim_{x \rightarrow 1} \frac{7x^2 - 7x}{x^3 - 5x^2 + 3x + 1} \\ &= \lim_{x \rightarrow 1} x + 2 + \lim_{x \rightarrow 1} \frac{7x(x - 1)}{x^3 - 5x^2 + 3x + 1} \end{aligned}$$

Dividing $x^3 - 5x^2 + 3x + 1$ by $x - 1$

$$\begin{array}{r} x^2 - 4x - 1 \\ x - 1 \overline{) x^3 - 5x^2 + 3x + 1} \\ \underline{+x^3 - x^2} \\ -4x^2 + 3x + 1 \\ \underline{-4x^2 + 4x} \\ -x + 1 \\ \underline{-x + 1} \\ \underline{\times} \end{array}$$

Q26

$$\lim_{x \rightarrow 2} \frac{x^3 + 3x^2 - 9x - 2}{x^3 - x - 6}$$

Divide $x^3 + 3x^2 - 9x - 2$ by $x^3 - x - 6$

$$\begin{array}{r} x^3 - x - 6 \overline{) x^3 + 3x^2 - 9x - 2} \\ \underline{+x^3 \quad -x \quad -6} \\ 3x^2 - 8x + 4 \end{array}$$

$$\begin{aligned} \Rightarrow \lim_{x \rightarrow 2} \frac{x^3 + 3x^2 - 9x - 2}{x^3 - x - 6} &= \lim_{x \rightarrow 2} 1 + \lim_{x \rightarrow 2} \frac{3x^2 - 8x + 4}{x^3 - x - 6} \\ &= 1 + \lim_{x \rightarrow 2} \frac{3x^2 - 2x - 6x + 4}{x^3 - x - 6} \\ &= 1 + \lim_{x \rightarrow 2} \frac{3x^2 - 2x - 6x + 4}{x^3 - x - 6} \\ \Rightarrow \lim_{x \rightarrow 2} \frac{x^3 + 3x^2 - 9x - 2}{x^3 - x - 6} &= 1 + \lim_{x \rightarrow 2} \frac{(3x - 2)(x - 2)}{x^3 - x - 6} \end{aligned}$$

Divide $x^3 - x - 6$ by $x - 2$

$$\begin{array}{r} x^2 + 2x + 3 \\ x - 2 \overline{) x^3 - x - 6} \\ \underline{+x^3 - 2x^2} \\ 2x^2 - x - 6 \\ \underline{2x^2 + 4x} \\ 3x - 6 \\ \underline{3x - 6} \\ 0 \end{array}$$

$$\begin{aligned} \Rightarrow \lim_{x \rightarrow 2} \frac{x^3 + 3x^2 - 9x - 2}{x^3 - x - 6} &= 1 + \lim_{x \rightarrow 2} \frac{(3x - 2)(x - 2)}{(x - 2)(x^2 + 2x + 3)} \\ &= 1 + \lim_{x \rightarrow 2} \frac{(3x - 2)}{x^2 + 2x + 3} \\ &= 1 + \frac{3 \times 2 - 2}{2^2 + 2 \times 2 + 3} \\ &= 1 + \frac{4}{11} \\ &= \frac{15}{11} \end{aligned}$$

Q27

$$\lim_{x \rightarrow 1} \frac{1 - x^{\frac{-1}{3}}}{1 - x^{\frac{-2}{3}}}$$

$$= \lim_{x \rightarrow 1} \frac{1 - \frac{1}{x^{\frac{1}{3}}}}{1 - \frac{1}{x^{\frac{2}{3}}}}$$

$$= \lim_{x \rightarrow 1} \frac{\left(x^{\frac{1}{3}} - 1\right)}{\left(x^{\frac{1}{3}} - 1\right)\left(x^{\frac{1}{3}} + 1\right)} \times x^{\frac{1}{3}}$$

$$= \lim_{x \rightarrow 1} \frac{x^{\frac{1}{3}}}{x^{\frac{1}{3}} + 1}$$

$$= \frac{1}{1 + 1}$$

$$= \frac{1}{2}$$

Q28

$$\lim_{x \rightarrow 3} \frac{x^2 - x - 6}{x^3 - 3x^2 + x - 3}$$

$$\begin{aligned}\text{Now } x^2 - x - 6 &= x^2 - 3x + 2x - 6 \\ &= x(x - 3) + 2(x - 3) \\ &= (x + 2)(x - 3) \quad \text{---(i)}\end{aligned}$$

Dividing $x^3 - 3x^2 + x - 3$ by $(x - 3)$, we get

$$\begin{array}{r} x^2 + 1 \\ x - 3 \overline{) x^3 - 3x^2 + x - 3} \\ \underline{\pm x^3 \mp 3x^2} \\ x - 3 \\ \underline{x - 3} \\ \underline{\times} \end{array}$$

Thus $(x - 3)$ is a factor of $x^3 - 3x^2 + x - 3$ ---(ii)

Substituting (i) and (ii) in the given expression

$$\begin{aligned}&= \lim_{x \rightarrow 3} \frac{(x + 2)(x - 3)}{(x^2 + 1)(x - 3)} \\ &= \frac{x + 2}{x^2 + 1} = \frac{3 + 2}{9 + 1} = \frac{5}{10} \\ &= \frac{1}{2}\end{aligned}$$

Q29

$$\lim_{x \rightarrow -2} \frac{x^3 + x^2 + 4x + 12}{x^3 - 3x + 2} = \lim_{x \rightarrow -2} \frac{(x+2)(x^2 - x + 6)}{(x+2)(x^2 - 2x + 1)}$$

$$\begin{aligned} & \lim_{x \rightarrow -2} \frac{x^2 - x + 6}{x^2 - 2x + 1} \\ &= \frac{(-2)^2 - (-2) + 6}{(-2)^2 - 2(-2) + 1} \\ &= \frac{4 + 2 + 6}{4 + 4 + 1} \\ &= \frac{12}{9} \\ &= \frac{4}{3} \end{aligned}$$

Q30

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{x^3 + 3x^2 - 6x + 2}{x^3 + 3x^2 - 3x - 1} &= \lim_{x \rightarrow 1} \frac{(x-1)(x^2 + 4x - 2)}{(x-1)(x^2 + 4x + 1)} \\ &= \frac{(1)^2 + 4(1) - 2}{(1)^2 + 4(1) + 1} = \frac{1 + 4 - 2}{1 + 4 + 1} \\ &= \frac{3}{6} \\ &= \frac{1}{2} \end{aligned}$$

Q31

$$\begin{aligned}& \lim_{x \rightarrow 2} \left[\frac{1}{x-2} - \frac{2(2x-3)}{x^3-3x^2+2x} \right] \\&= \lim_{x \rightarrow 2} \left[\frac{1}{x-2} - \frac{2(2x-3)}{x(x^2-2x-x+2)} \right] \\&= \lim_{x \rightarrow 2} \left[\frac{1}{x-2} - \frac{2(2x-3)}{x(x-2)(x-1)} \right] \\&= \lim_{x \rightarrow 2} \left[\frac{x(x-1)-2(2x-3)}{x(x-1)(x-2)} \right] \\&= \lim_{x \rightarrow 2} \left[\frac{x^2-x-4x+6}{x(x-1)(x-2)} \right] \\&= \lim_{x \rightarrow 2} \left[\frac{x^2-2x-3x+6}{x(x-1)(x-2)} \right] \\&= \lim_{x \rightarrow 2} \left[\frac{x(x-2)-3(x-2)}{x(x-1)(x-2)} \right] \\&= \lim_{x \rightarrow 2} \left[\frac{(x-2)(x-3)}{x(x-1)(x-2)} \right] \\&= \lim_{x \rightarrow 2} \left[\frac{1}{x-2} - \frac{2(2x-3)}{x(x-2)(x-1)} \right] \\&= \frac{-1}{2}\end{aligned}$$

Q32

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{\sqrt{x^2 - 1} + \sqrt{x - 1}}{\sqrt{x^2 - 1}} \\ &= \lim_{x \rightarrow 1} \frac{(\sqrt{x^2 - 1} + \sqrt{x - 1})}{\sqrt{x^2 - 1}} \times \frac{(\sqrt{x^2 - 1} - \sqrt{x - 1})}{(\sqrt{x^2 - 1} - \sqrt{x - 1})} \times \frac{\sqrt{x^2 - 1}}{\sqrt{x^2 - 1}} \\ &= \lim_{x \rightarrow 1} \frac{[(x^2 - 1) - (x - 1)] \times \sqrt{x^2 - 1}}{(x^2 - 1)(\sqrt{x^2 - 1} - \sqrt{x - 1})} \\ &= \lim_{x \rightarrow 1} \frac{(x^2 - x) \sqrt{x^2 - 1}}{(x^2 - 1)(\sqrt{x^2 - 1} - \sqrt{x - 1})} \\ &= \lim_{x \rightarrow 1} \frac{x(x - 1) \sqrt{x^2 - 1}}{(x - 1)(x + 1)(\sqrt{x^2 - 1} - \sqrt{x - 1})} \\ &= \lim_{x \rightarrow 1} \frac{x \cancel{(\sqrt{x - 1})} (\sqrt{x + 1})}{(x + 1) \cancel{(\sqrt{x - 1})} (\sqrt{x + 1} - 1)} \\ &= \frac{\sqrt{2}}{2(\sqrt{2} - 1)} \\ &= \frac{\sqrt{2}}{2 \times (\sqrt{2} - 1)} \times \frac{\sqrt{2} + 1}{\sqrt{2} + 1} \\ &= \frac{\sqrt{2} + 1}{\sqrt{2}} \end{aligned}$$

Q33

$$\begin{aligned}
& \lim_{x \rightarrow 1} \left\{ \frac{x-2}{x^2-x} - \frac{1}{x^3-3x^2+2x} \right\} \\
&= \lim_{x \rightarrow 1} \left\{ \frac{x-2}{x(x-1)} - \frac{1}{x(x^2-3x+2)} \right\} \\
&= \lim_{x \rightarrow 1} \left\{ \frac{x-2}{x(x-1)} - \frac{1}{x(x^2-3x+2)} \right\} \\
&= \lim_{x \rightarrow 1} \left\{ \frac{x-2}{x(x-1)} - \frac{1}{x(x^2-1x-2x+2)} \right\} \\
&= \lim_{x \rightarrow 1} \left\{ \frac{x-2}{x(x-1)} - \frac{1}{x(x-1)(x-2)} \right\} \\
&= \lim_{x \rightarrow 1} \left\{ \frac{(x-2)^2-1}{x(x-1)(x-2)} \right\} \\
&= \lim_{x \rightarrow 1} \left\{ \frac{x^2+4-4x-1}{x(x-1)(x-2)} \right\} \\
&= \lim_{x \rightarrow 1} \left\{ \frac{x^2-4x+3}{x(x-1)(x-2)} \right\} \\
&= \lim_{x \rightarrow 1} \left[\frac{x^2-x-3x+3}{x(x-1)(x-2)} \right] \\
&= \lim_{x \rightarrow 1} \left[\frac{x(x-1)-3(x-1)}{x(x-1)(x-2)} \right] \\
&= \lim_{x \rightarrow 1} \left[\frac{(x-3)(x-1)}{x(x-1)(x-2)} \right] \\
&= \frac{(1-3)}{1(1-2)} \\
&= \frac{-2}{-1} \\
&= 2
\end{aligned}$$

Q34

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{x^7 - 2x^5 + 1}{x^3 - 3x^2 + 2} &= \lim_{x \rightarrow 1} \frac{(x-1)(x^6 + x^5 - x^4 - x^3 - x^2 - x - 1)}{(x-1)(x^2 - 2x - 2)} \\&= \lim_{x \rightarrow 1} \frac{(x^6 + x^5 - x^4 - x^3 - x^2 - x - 1)}{(x^2 - 2x - 2)} \\&= \frac{(1 + 1 - 1 - 1 - 1 - 1 - 1)}{(1 - 2 - 2)} \\&= \frac{-3}{-3} \\&= 1\end{aligned}$$

Ex 29.4

Q1

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sqrt{1+x+x^2} - 1}{x} &= \lim_{x \rightarrow 0} \frac{\left(\sqrt{1+x+x^2} - 1\right) \left(\sqrt{1+x+x^2} + 1\right)}{x \left(\sqrt{1+x+x^2} + 1\right)} \\&= \lim_{x \rightarrow 0} \frac{\left(\left(1+x+x^2\right) - 1\right)}{x \left(\sqrt{1+x+x^2} + 1\right)} \\&= \lim_{x \rightarrow 0} \frac{x(1+x)}{x \left(\sqrt{1+x+x^2} + 1\right)} \\&= \lim_{x \rightarrow 0} \frac{1+x}{\sqrt{1+x+x^2} + 1} \\&= \frac{1+0}{\sqrt{1+0+0} + 1} \\&= \frac{1}{1+1} \\&= \frac{1}{2}\end{aligned}$$

Q2

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{2x}{\sqrt{a+x} - \sqrt{a-x}} &= \lim_{x \rightarrow 0} \frac{2x}{\left(\sqrt{a+x} - \sqrt{a-x}\right)} \times \frac{\sqrt{a+x} + \sqrt{a-x}}{\sqrt{a+x} + \sqrt{a-x}} \\&= \lim_{x \rightarrow 0} \frac{2x \left(\sqrt{a+x} + \sqrt{a-x}\right)}{\left(\left(a+x\right) - \left(a-x\right)\right)} \\&= \lim_{x \rightarrow 0} \frac{2x \left(\sqrt{a+x} + \sqrt{a-x}\right)}{2x} \\&= \lim_{x \rightarrow 0} \left(\sqrt{a+x} + \sqrt{a-x}\right) \\&= \sqrt{a} + \sqrt{a} \\&= 2\sqrt{a}\end{aligned}$$

Q3

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{\sqrt{a^2 + x^2} - a}{x^2} \\ &= \lim_{x \rightarrow 0} \frac{(\sqrt{a^2 + x^2} - a)}{x^2} \times \frac{(\sqrt{a^2 + x^2} + a)}{(\sqrt{a^2 + x^2} + a)} \\ &= \lim_{x \rightarrow 0} \frac{(a^2 + x^2 - a^2)}{x^2 \sqrt{a^2 + x^2} + a} \\ &= \lim_{x \rightarrow 0} \frac{x^2}{x^2 (\sqrt{a^2 + x^2} + a)} \\ &= \lim_{x \rightarrow 0} \frac{1}{\sqrt{a^2 + x^2} + a} \\ &= \frac{1}{a + a} \\ &= \frac{1}{2a} \end{aligned}$$

Q4

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{2x} \\ &= \lim_{x \rightarrow 0} \frac{(\sqrt{1+x} - \sqrt{1-x})}{2x} \times \frac{(\sqrt{1+x} + \sqrt{1-x})}{(\sqrt{1+x} + \sqrt{1-x})} \\ &= \lim_{x \rightarrow 0} \frac{(1+x) - (1-x)}{2x (\sqrt{1+x} + \sqrt{1-x})} \\ &= \lim_{x \rightarrow 0} \frac{2x}{2x (\sqrt{1+x} + \sqrt{1-x})} \\ &= \lim_{x \rightarrow 0} \frac{1}{(\sqrt{1+x} + \sqrt{1-x})} \\ &= \frac{1}{\sqrt{1} + \sqrt{1}} \\ &= \frac{1}{2} \end{aligned}$$

Q5

$$\begin{aligned} & \lim_{x \rightarrow 2} \frac{\sqrt{3-x} - 1}{2-x} \\ &= \lim_{x \rightarrow 2} \frac{(\sqrt{3-x} - 1)(\sqrt{3-x} + 1)}{(2-x)(\sqrt{3-x} + 1)} \\ &= \lim_{x \rightarrow 2} \frac{(3-x) - 1}{(2-x)(\sqrt{3-x} + 1)} \\ &= \lim_{x \rightarrow 2} \frac{(2-x)}{(2-x)(\sqrt{3-x} + 1)} \\ &= \lim_{x \rightarrow 2} \frac{1}{(\sqrt{3-x} + 1)} \\ &= \frac{1}{\sqrt{3-2} + 1} = \frac{1}{1+1} \\ &= \frac{1}{2} \end{aligned}$$

Q6

$$\begin{aligned} & \lim_{x \rightarrow 3} \frac{x-3}{\sqrt{x-2} - \sqrt{4-x}} \\ &= \lim_{x \rightarrow 3} \frac{x-3}{(\sqrt{x-2} - \sqrt{4-x})} \times \frac{(\sqrt{x-2} + \sqrt{4-x})}{(\sqrt{x-2} + \sqrt{4-x})} \\ &= \lim_{x \rightarrow 3} \frac{(x-3)(\sqrt{x-2} + \sqrt{4-x})}{(x-2) - (4-x)} \\ &= \lim_{x \rightarrow 3} \frac{(x-3)(\sqrt{x-2} + \sqrt{4-x})}{x-2-4+x} \\ &= \lim_{x \rightarrow 3} \frac{(x-3)(\sqrt{x-2} + \sqrt{4-x})}{2(x-3)} \\ &= \frac{1}{2} \lim_{x \rightarrow 3} (\sqrt{x-2} + \sqrt{4-x}) \\ &= \frac{1}{2} (\sqrt{3-2} + \sqrt{4-3}) \\ &= \frac{1}{2} (\sqrt{1} + \sqrt{1}) \\ &= \frac{1}{2} (1+1) = \frac{2}{2} \\ &= 1 \end{aligned}$$

Q7

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{x}{\left(\sqrt{1+x} - \sqrt{1-x}\right)} \\&= \lim_{x \rightarrow 0} \frac{x}{\left(\sqrt{1+x} - \sqrt{1-x}\right)} \times \frac{\left(\sqrt{1+x} + \sqrt{1-x}\right)}{\left(\sqrt{1+x} + \sqrt{1-x}\right)} \\&= \lim_{x \rightarrow 0} \frac{x \left(\sqrt{1+x} + \sqrt{1-x}\right)}{\left(\sqrt{1+x}\right)^2 - \left(\sqrt{1-x}\right)^2} \\&= \lim_{x \rightarrow 0} \frac{x \left(\sqrt{1+x} + \sqrt{1-x}\right)}{1+x-1+x} \\&= \lim_{x \rightarrow 0} \frac{x \left(\sqrt{1+x} + \sqrt{1-x}\right)}{2x} \\&= \frac{1}{2} \lim_{x \rightarrow 0} \left(\frac{\sqrt{1+x} + \sqrt{1-x}}{x} \right) x \\&= \frac{1}{2} \lim_{x \rightarrow 0} \left(\sqrt{1+x} + \sqrt{1-x} \right) \\&= \frac{1}{2} \left(\sqrt{1} + \sqrt{1} \right) \\&= \frac{2}{2} = 1\end{aligned}$$

Q8

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{\left(\sqrt{5x-4} - \sqrt{x}\right)}{x-1} \\&= \lim_{x \rightarrow 1} \frac{\left(\sqrt{5x-4} - \sqrt{x}\right)}{x-1} \times \frac{\left(\sqrt{5x-4} + \sqrt{x}\right)}{\left(\sqrt{5x-4} + \sqrt{x}\right)} \\&= \lim_{x \rightarrow 1} \frac{\left((5x-4) - x\right)}{(x-1) \left(\sqrt{5x-4} + \sqrt{x}\right)} \\&= 4 \lim_{x \rightarrow 1} \frac{(x-1)}{(x-1) \left(\sqrt{5x-4} + \sqrt{x}\right)} \\&= 4 \lim_{x \rightarrow 1} \frac{1}{\sqrt{5x-4} + \sqrt{x}} \\&= 4 \times \frac{1}{\sqrt{5-4} + \sqrt{1}} \\&= 4 \times \frac{1}{\sqrt{1} + \sqrt{1}} \\&= \frac{4}{2} = 2\end{aligned}$$

Q9

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{(x-1)}{(\sqrt{x^2+3}-2)} &= \lim_{x \rightarrow 1} \frac{(x-1) \times (\sqrt{x^2+3}+2)}{(\sqrt{x^2+3}-2)(\sqrt{x^2+3}+2)} \\&= \lim_{x \rightarrow 1} \frac{(x-1)(\sqrt{x^2+3}+2)}{(x^2+3-4)} \\&= \lim_{x \rightarrow 1} \frac{(x-1)(\sqrt{x^2+3}+2)}{(x^2-1)} \\&= \lim_{x \rightarrow 1} \frac{\sqrt{x^2+3}+2}{x+1}\end{aligned}$$

Putting the value $x = 1$

$$\begin{aligned}\Rightarrow & \frac{\sqrt{1+3}+2}{1+1} \\&= \frac{2+2}{2} \\&= \frac{4}{2} = 2\end{aligned}$$

Q10

$$\begin{aligned}\lim_{x \rightarrow 3} \frac{(\sqrt{x+3} - \sqrt{6})}{(x^2 - 9)} \\&= \lim_{x \rightarrow 3} \frac{(\sqrt{x+3} - \sqrt{6}) (\sqrt{x+3} - \sqrt{6})}{(x-3)(x+3)(\sqrt{x+3} + \sqrt{6})} \\&= \lim_{x \rightarrow 3} \frac{((x+3) - 6)}{(x-3)(x+3)(\sqrt{x+3} + \sqrt{6})} \\&= \lim_{x \rightarrow 3} \frac{(x-3)}{(x-3)(x+3)(\sqrt{x+3} + \sqrt{6})} \\&= \lim_{x \rightarrow 3} \frac{1}{(x+3)(\sqrt{x+3} + \sqrt{6})} \\&= \frac{1}{(3+3)\sqrt{3+3} + \sqrt{6}} \\&= \frac{1}{6(\sqrt{6} + \sqrt{6})} = \frac{1}{6 \times 2\sqrt{6}} \\&= \frac{1}{12\sqrt{6}}\end{aligned}$$

Q11

$$\begin{aligned}
& \lim_{x \rightarrow 1} \frac{(\sqrt{5x-4} - \sqrt{x})}{(x^2 - 1)} \\
&= \lim_{x \rightarrow 1} \frac{(\sqrt{5x-4} - \sqrt{x})}{(x-1)(x+1)} \times \frac{(\sqrt{5x-4} + \sqrt{x})}{(\sqrt{5x-4} + \sqrt{x})} \\
&= \lim_{x \rightarrow 1} \frac{((5x-4) - x)}{(x-1)(x+1)(\sqrt{5x-4} + \sqrt{x})} \\
&= \lim_{x \rightarrow 1} \frac{4(x-1)}{(x-1)(x+1)(\sqrt{5x-4} + \sqrt{x})} \\
&= \lim_{x \rightarrow 1} \frac{4}{(x+1)(\sqrt{5x-4} + \sqrt{x})} \\
&= \frac{4}{(1+1)(\sqrt{5-4} + \sqrt{1})} \\
&= \frac{4}{2(1+1)} \\
&= \frac{4}{4} = 1
\end{aligned}$$

Q12

$$\begin{aligned}
& \lim_{x \rightarrow 0} \frac{(\sqrt{1+x} - 1)}{x} \\
&= \lim_{x \rightarrow 0} \frac{(\sqrt{1+x} - 1)}{x} \times \frac{(\sqrt{1+x} + 1)}{(\sqrt{1+x} + 1)} \\
&= \lim_{x \rightarrow 0} \frac{(1+x-1)}{x(\sqrt{1+x} + 1)} \\
&= \lim_{x \rightarrow 0} \frac{x}{x(\sqrt{1+x} + 1)} \\
&= \lim_{x \rightarrow 0} \frac{1}{(\sqrt{1+x} + 1)} \\
&= \frac{1}{\sqrt{1+1}} = \frac{1}{2}
\end{aligned}$$

Q13

$$\begin{aligned}
& \lim_{x \rightarrow 2} \frac{\sqrt{x^2 + 1} - \sqrt{5}}{(x - 2)} \\
&= \lim_{x \rightarrow 2} \frac{\sqrt{x^2 + 1} - \sqrt{5}}{(x - 2)} \times \frac{(\sqrt{x^2 + 1} + \sqrt{5})}{(\sqrt{x^2 + 1} + \sqrt{5})} \\
&= \lim_{x \rightarrow 2} \frac{(x^2 + 1 - 5)}{(x - 2)(\sqrt{x^2 + 1} + \sqrt{5})} \\
&= \lim_{x \rightarrow 2} \frac{(x + 2)(x - 2)}{(x - 2)(\sqrt{x^2 + 1} + \sqrt{5})} \\
&= \lim_{x \rightarrow 2} \frac{(x + 2)}{(\sqrt{x^2 + 1} + \sqrt{5})} \\
&= \frac{(2 + 2)}{\sqrt{4 + 1} + \sqrt{5}} \\
&= \frac{4}{2\sqrt{5}} = \frac{2}{\sqrt{5}}
\end{aligned}$$

Q14

$$\begin{aligned}
& \lim_{x \rightarrow 2} \frac{x - 2}{\sqrt{x} - \sqrt{2}} \\
&= \lim_{x \rightarrow 2} \frac{(x - 2)(\sqrt{x} + \sqrt{2})}{(\sqrt{x} - \sqrt{2})(\sqrt{x} + \sqrt{2})} \\
&= \lim_{x \rightarrow 2} \frac{(x - 2)(\sqrt{x} + \sqrt{2})}{(x - 2)} \\
&= \sqrt{2} + \sqrt{2} \\
&= 2\sqrt{2}
\end{aligned}$$

Q15

$$\begin{aligned}
& \lim_{x \rightarrow 7} \frac{4 - \sqrt{9+x}}{1 - \sqrt{8-x}} \\
&= \lim_{x \rightarrow 7} \frac{(4 - \sqrt{9+x})}{1 - \sqrt{8-x}} \times \frac{(4 + \sqrt{9+x})}{(4 + \sqrt{9+x})} \times \frac{(1 + \sqrt{8-x})}{(1 + \sqrt{8-x})} \\
&= \lim_{x \rightarrow 7} \frac{\left((4)^2 - (\sqrt{9+x})^2\right)}{\left((1)^2 - (\sqrt{8-x})^2\right)} \times \frac{1 + \sqrt{8-x}}{4 + \sqrt{9+x}} \\
&= \lim_{x \rightarrow 7} \frac{(16 - 9 - x) \times (1 + \sqrt{8-x})}{(1 - 8 + x) \times (4 + \sqrt{9+x})} \\
&= \lim_{x \rightarrow 7} \frac{7 - x}{(-1)(7 - x)} \times \frac{(1 + \sqrt{8-x})}{(4 + \sqrt{9+x})} \\
&= \frac{1}{(-1)} \times \frac{(1+1)}{(4+4)} = \frac{-1}{4}
\end{aligned}$$

Q16

$$\begin{aligned}
& \lim_{x \rightarrow 0} \frac{\sqrt{a+x} - \sqrt{a}}{x\sqrt{a^2+ax}} \\
&= \lim_{x \rightarrow 0} \frac{(\sqrt{a+x} - \sqrt{a})}{x\sqrt{a^2+ax}} \times \frac{(\sqrt{a+x} + \sqrt{a})}{(\sqrt{a+x} + \sqrt{a})} \\
&= \lim_{x \rightarrow 0} \frac{(a+x) - a}{x\sqrt{a^2+ax}(\sqrt{a+x} + \sqrt{a})} \\
&= \lim_{x \rightarrow 0} \frac{x}{x\sqrt{a^2+ax}(\sqrt{a+x} + \sqrt{a})} \\
&= \lim_{x \rightarrow 0} \frac{1}{(\sqrt{a^2+ax})(\sqrt{a+x} + \sqrt{a})} \\
&= \frac{1}{a(2\sqrt{a})} \\
&= \frac{1}{2a\sqrt{a}}
\end{aligned}$$

Q17

$$\lim_{x \rightarrow 5} \frac{x - 5}{\sqrt{6x - 5} - \sqrt{4x + 5}}$$

$$= \lim_{x \rightarrow 5} \frac{x - 5}{\sqrt{6x - 5} - \sqrt{4x + 5}} \times \frac{(\sqrt{6x - 5} + \sqrt{4x + 5})}{(\sqrt{6x - 5} + \sqrt{4x + 5})}$$

$$= \lim_{x \rightarrow 5} \frac{(x - 5)(\sqrt{6x - 5} + \sqrt{4x + 5})}{(6x - 5) - (4x + 5)}$$

$$= \lim_{x \rightarrow 5} \frac{(x - 5)(\sqrt{6x - 5} + \sqrt{4x + 5})}{2x - 10}$$

$$= \lim_{x \rightarrow 5} \frac{(x - 5)(\sqrt{6x - 5} + \sqrt{4x + 5})}{2(x - 5)}$$

$$= \frac{\sqrt{6(5) - 5} + \sqrt{4(5) + 5}}{2}$$

$$= \frac{\sqrt{25} + \sqrt{25}}{2}$$

$$= \frac{5 + 5}{2} = 5$$

Q18

$$\begin{aligned}
& \lim_{x \rightarrow 1} \frac{\sqrt{5x-4} - \sqrt{x}}{x^3 - 1} \\
&= \lim_{x \rightarrow 1} \frac{(\sqrt{5x-4} - \sqrt{x})}{(x-1)(x^2+1+1)} \times \frac{(\sqrt{5x-4} + \sqrt{x})}{(\sqrt{5x-4} + \sqrt{x})} \\
&= \lim_{x \rightarrow 1} \frac{(5x-4-x)}{(x-1)(x^2+1+x)(\sqrt{5x-4} + \sqrt{x})} \\
&= \lim_{x \rightarrow 1} \frac{4(x-1)}{(x-1)(x^2+x+1)(\sqrt{5x-4} + \sqrt{x})} \\
&= \frac{4}{(1+1+1)(\sqrt{5-4} + \sqrt{1})} \\
&= \frac{4}{(3)(1+1)} \\
&= \frac{4}{3 \times 2} = \frac{2}{3}
\end{aligned}$$

Q19

$$\begin{aligned}
& \lim_{x \rightarrow 2} \frac{\sqrt{1+4x} - \sqrt{5+2x}}{x-2} \\
&= \lim_{x \rightarrow 2} \frac{(\sqrt{1+4x} - \sqrt{5+2x})}{(x-2)} \times \frac{(\sqrt{1+4x} + \sqrt{5+2x})}{(\sqrt{1+4x} + \sqrt{5+2x})} \\
&= \lim_{x \rightarrow 2} \frac{(1+4x) - (5+2x)}{(x-2)(\sqrt{1+4x} + \sqrt{5+2x})} \\
&= \lim_{x \rightarrow 2} \frac{-4+2x}{(x-2)(\sqrt{1+4x} + \sqrt{5+2x})} \\
&= \lim_{x \rightarrow 2} \frac{2(x-2)}{(x-2)(\sqrt{1+4x} + \sqrt{5+2x})} \\
&= \frac{2}{\sqrt{1+8} + \sqrt{5+4}} = \frac{2}{\sqrt{9} + \sqrt{9}} \\
&= \frac{2}{6} = \frac{1}{3}
\end{aligned}$$

Q20

$$\lim_{x \rightarrow 1} \frac{\sqrt{3+x} - \sqrt{5-x}}{x^2 - 1}$$

$$= \lim_{x \rightarrow 1} \frac{(\sqrt{3+x} - \sqrt{5-x})}{(x^2 - 1)} \times \frac{(\sqrt{3+x} + \sqrt{5-x})}{(\sqrt{3+x} + \sqrt{5-x})}$$

$$= \lim_{x \rightarrow 1} \frac{(3+x) - (5-x)}{(x-1)(x+1)(\sqrt{3+x} + \sqrt{5-x})}$$

$$= \lim_{x \rightarrow 1} \frac{-2+2x}{(x-1)(x+1)(\sqrt{3+x} + \sqrt{5-x})}$$

$$= \lim_{x \rightarrow 1} \frac{2(x-1)}{(x-1)(x+1)(\sqrt{3+x} + \sqrt{5-x})}$$

$$= \lim_{x \rightarrow 1} \frac{2}{(x+1)(\sqrt{3+x} + \sqrt{5-x})}$$

$$= \frac{2}{(1+1)(\sqrt{3+1} + \sqrt{5-1})} = \frac{2}{(2)(2+2)}$$

$$= \frac{1}{4}$$

Q21

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{x} \\&= \lim_{x \rightarrow 0} \frac{\left(\sqrt{1+x^2} - \sqrt{1-x^2}\right)}{x} \times \frac{\left(\sqrt{1+x^2} + \sqrt{1-x^2}\right)}{\left(\sqrt{1+x^2} + \sqrt{1-x^2}\right)} \\&= \lim_{x \rightarrow 0} \frac{(1+x^2) - (1-x^2)}{x \left(\sqrt{1+x^2} + \sqrt{1-x^2}\right)} \\&= \lim_{x \rightarrow 0} \frac{2x^2}{x \left(\sqrt{1+x^2} + \sqrt{1-x^2}\right)} \\&= \frac{2 \times 0}{\left(\sqrt{1} + \sqrt{1}\right)} \\&= \frac{2}{2} \times 0 \\&= 0\end{aligned}$$

Q22

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sqrt{1+x+x^2} - \sqrt{x+1}}{2x^2} \\&= \lim_{x \rightarrow 0} \frac{\left(\sqrt{1+x+x^2} - \sqrt{x+1}\right) \times \left(\sqrt{1+x+x^2} + \sqrt{x+1}\right)}{2x^2 \left(\sqrt{1+x+x^2} + \sqrt{x+1}\right)} \\&= \lim_{x \rightarrow 0} \frac{(1+x+x^2) - (x+1)}{2x^2 \left(\sqrt{1+x+x^2} + \sqrt{x+1}\right)} \\&= \lim_{x \rightarrow 0} \frac{x^2}{2x^2 \left(\sqrt{1+x+x^2} + \sqrt{x+1}\right)} \\&= \frac{1}{2 \left(\sqrt{1} + \sqrt{1}\right)} \\&= \frac{1}{2 \times 2} \\&= \frac{1}{4}\end{aligned}$$

Q23

$$\begin{aligned}\lim_{x \rightarrow 4} \frac{2 - \sqrt{x}}{4 - x} \\&= \lim_{x \rightarrow 4} \frac{(2 - \sqrt{x})(2 + \sqrt{x})}{(4 - x)(2 + \sqrt{x})} \\&= \lim_{x \rightarrow 4} \frac{(2)^2 - (\sqrt{x})^2}{(4 - x)(2 + \sqrt{x})} \\&= \lim_{x \rightarrow 4} \frac{(4 - x)}{(4 - x)(2 + \sqrt{x})} \\&= \frac{1}{2 + \sqrt{4}} \\&= \frac{1}{2 + 2} \\&= \frac{1}{4}\end{aligned}$$

Q24

$$\begin{aligned}\lim_{x \rightarrow a} \frac{x - a}{\sqrt{x} - \sqrt{a}} \\&= \lim_{x \rightarrow a} \frac{(x - a)(\sqrt{x} + \sqrt{a})}{(\sqrt{x} - \sqrt{a})(\sqrt{x} + \sqrt{a})} \\&= \lim_{x \rightarrow a} \frac{(x - a)(\sqrt{x} + \sqrt{a})}{(x - a)} \\&= \lim_{x \rightarrow a} (\sqrt{x} + \sqrt{a}) \\&= 2\sqrt{a}\end{aligned}$$

Q25

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sqrt{1+3x} - \sqrt{1-3x}}{x} \\&= \lim_{x \rightarrow 0} \frac{(\sqrt{1+3x} - \sqrt{1-3x})}{x} \times \frac{(\sqrt{1+3x} + \sqrt{1-3x})}{(\sqrt{1+3x} + \sqrt{1-3x})} \\&= \lim_{x \rightarrow 0} \frac{(1+3x) - (1-3x)}{x(\sqrt{1+3x} + \sqrt{1-3x})} \\&= \lim_{x \rightarrow 0} \frac{6x}{x(\sqrt{1+3x} + \sqrt{1-3x})} \\&= \lim_{x \rightarrow 0} \frac{6}{(\sqrt{1+3x} + \sqrt{1-3x})} \\&= \frac{6}{\sqrt{1} + \sqrt{1}} \\&= \frac{6}{2} \\&= 3\end{aligned}$$

Q26

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{\sqrt{2-x} - \sqrt{2+x}}{x} &= \lim_{x \rightarrow 0} \frac{(\sqrt{2-x} + \sqrt{2+x}) \times (\sqrt{2-x} - \sqrt{2+x})}{x \times (\sqrt{2-x} + \sqrt{2+x})} \\
 &= \lim_{x \rightarrow 0} \frac{(2-x) - (2+x)}{x(\sqrt{2-x} + \sqrt{2+x})} \\
 &= \lim_{x \rightarrow 0} \frac{-2x}{x(\sqrt{2-x} + \sqrt{2+x})} \\
 &= \frac{-2}{\sqrt{2} + \sqrt{2}} \\
 &= \frac{-2}{2\sqrt{2}} \\
 &= \frac{-1}{\sqrt{2}}
 \end{aligned}$$

Q27

$$\begin{aligned}
 \lim_{x \rightarrow 1} \frac{\sqrt{3+x} - \sqrt{5-x}}{x^2 - 1} &= \lim_{x \rightarrow 1} \frac{(\sqrt{3+x} - \sqrt{5-x}) \times (\sqrt{3+x} + \sqrt{5-x})}{(x-1)(x+1) \times (\sqrt{3+x} + \sqrt{5-x})} \\
 &= \lim_{x \rightarrow 1} \frac{((3+x) - (5-x))}{(x-1)(x+1)(\sqrt{3+x} + \sqrt{5-x})} \\
 &= \lim_{x \rightarrow 1} \frac{-2 + 2x}{(x-1)(x+1)(\sqrt{3+x} + \sqrt{5-x})} \\
 &= \lim_{x \rightarrow 1} \frac{-2(x-1)}{(x-1)(x+1)(\sqrt{3+x} + \sqrt{5-x})} \\
 &= \lim_{x \rightarrow 1} \frac{-2}{(x+1)(\sqrt{3+x} + \sqrt{5-x})} \\
 &= \frac{2}{(1+1)(\sqrt{3+1} + \sqrt{5-1})} \\
 &= \frac{2}{(2)(2+2)} \\
 &= \frac{1}{4}
 \end{aligned}$$

Q28

$$\begin{aligned}
 \lim_{x \rightarrow 1} \frac{(2x-3)(\sqrt{x}-1)}{3x^2+3x-6} &= \lim_{x \rightarrow 1} \frac{(2x-3)(\sqrt{x}-1)}{3[x^2+x-2]} \\
 &= \lim_{x \rightarrow 1} \frac{(2x-3)(\sqrt{x}-1)}{3[x^2+2x-x-2]} \\
 &= \lim_{x \rightarrow 1} \frac{(2x-3)(\sqrt{x}-1)}{3(x+2)(x-1)} \\
 &= \lim_{x \rightarrow 1} \frac{(2x-3)(\sqrt{x}-1)}{3(x+2)(\sqrt{x}-1)(\sqrt{x}+1)} \\
 &= \lim_{x \rightarrow 1} \frac{(2x-3)}{3(x+2)(\sqrt{x}+1)} \\
 &= \frac{(2-3)}{3(1+2)(1+1)} \\
 &= \frac{-1}{3 \times 3 \times 2} \\
 &= -\frac{1}{18}
 \end{aligned}$$

Q29

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{\sqrt{1+x^2} - \sqrt{1+x}}{\sqrt{1+x^3} - \sqrt{1+x}} &= \lim_{x \rightarrow 0} \frac{\left(\sqrt{1+x^2} - \sqrt{1+x}\right)}{\left(\sqrt{1+x^3} - \sqrt{1+x}\right)} \times \frac{\left(\sqrt{1+x^2} + \sqrt{1+x}\right)}{\left(\sqrt{1+x^2} + \sqrt{1+x}\right)} \\
 &= \lim_{x \rightarrow 0} \frac{(1+x^2) - (1+x) \times \left(\sqrt{1+x^3} + \sqrt{1+x}\right)}{\left(\sqrt{1+x^3} - \sqrt{1+x}\right) \left(\sqrt{1+x^2} + \sqrt{1+x}\right) \left(\sqrt{1+x^3} + \sqrt{1+x}\right)} \\
 &= \lim_{x \rightarrow 0} \frac{(x^2 - x) \left(\sqrt{1+x^3} + \sqrt{1+x}\right)}{\left(\sqrt{1+x^2} + \sqrt{1+x}\right) (1+x^3 - 1 - x)} \\
 &= \lim_{x \rightarrow 0} \frac{x(x-1) \left(\sqrt{1+x^3} + \sqrt{1+x}\right)}{\left(\sqrt{1+x^2} + \sqrt{1+x}\right) x(x^2 - 1)} \\
 &= \lim_{x \rightarrow 0} \frac{x(x-1) \left(\sqrt{1+x^3} + \sqrt{1+x}\right)}{\left(\sqrt{1+x^2} + \sqrt{1+x}\right) (x)(x-1)(x+1)} \\
 &= \lim_{x \rightarrow 0} \frac{\left(\sqrt{1+x^3} + \sqrt{1+x}\right)}{\left(\left(\sqrt{1+x^2} + \sqrt{1+x}\right)(x+1)\right)} \\
 &= \frac{2}{2} = 1
 \end{aligned}$$

Q30

$$\begin{aligned}
& \lim_{x \rightarrow 1} \frac{x^2 - \sqrt{x}}{\sqrt{x} - 1} \\
&= \lim_{x \rightarrow 1} \frac{(x^2 - \sqrt{x})(x^2 + \sqrt{x})}{(\sqrt{x} - 1)(x^2 + \sqrt{x})} \\
&= \lim_{x \rightarrow 1} \frac{x^4 - x}{(\sqrt{x} - 1)(x^2 + \sqrt{x})} \\
&= \lim_{x \rightarrow 1} \frac{x(x^3 - 1)}{(\sqrt{x} - 1)(x^2 + \sqrt{x})} \\
&= \lim_{x \rightarrow 1} \frac{x(x-1)(x^2 + 1 + x)}{(\sqrt{x} - 1)(x^2 + \sqrt{x})} \\
&= \lim_{x \rightarrow 1} \frac{x(\sqrt{x} - 1)(\sqrt{x} + 1)(x^2 + 1 + x)}{(\sqrt{x} - 1)(x^2 + \sqrt{x})} \\
&= \lim_{x \rightarrow 1} \frac{x(\sqrt{x} + 1)(x^2 + 1 + x)}{(x^2 + \sqrt{x})} \\
&= \frac{1(1+1)(1+1+1)}{1+1} \\
&= \frac{6}{2} \\
&= 3
\end{aligned}$$

Q31

$$\begin{aligned}
& \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
&= \lim_{h \rightarrow 0} \frac{(\sqrt{x+h} - \sqrt{x})}{h} \times \frac{(\sqrt{x+h} + \sqrt{x})}{(\sqrt{x+h} + \sqrt{x})} \\
&= \lim_{h \rightarrow 0} \frac{(x+h-x)}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \frac{1}{\sqrt{x} + \sqrt{x}} \\
&= \frac{1}{2\sqrt{x}}
\end{aligned}$$

Q32

$$\begin{aligned}
 & \lim_{x \rightarrow \sqrt{10}} \frac{\sqrt{7+2x} - (\sqrt{5} + \sqrt{2})}{x^2 - 10} \\
 &= \lim_{x \rightarrow \sqrt{10}} \frac{\sqrt{7+2x} - \left(\sqrt{(\sqrt{5} + \sqrt{2})^2} \right)}{x^2 - 10} \\
 &= \lim_{x \rightarrow \sqrt{10}} \frac{\sqrt{7+2x} - (\sqrt{7+2\sqrt{10}})}{x^2 - 10} \\
 &= \lim_{x \rightarrow \sqrt{10}} \frac{\sqrt{7+2x} - (\sqrt{7+2\sqrt{10}})}{x^2 - 10} \times \frac{\sqrt{7+2x} + (\sqrt{7+2\sqrt{10}})}{\sqrt{7+2x} + (\sqrt{7+2\sqrt{10}})} \\
 &= \lim_{x \rightarrow \sqrt{10}} \frac{7+2x - 7 - 2\sqrt{10}}{(x^2 - 10)(\sqrt{7+2x} + (\sqrt{7+2\sqrt{10}}))} \\
 &= \lim_{x \rightarrow \sqrt{10}} \frac{2(x - \sqrt{10})}{(x^2 - 10)(\sqrt{7+2x} + (\sqrt{7+2\sqrt{10}}))} \\
 &= \lim_{x \rightarrow \sqrt{10}} \frac{2}{(x + \sqrt{10})(\sqrt{7+2x} + (\sqrt{7+2\sqrt{10}}))} \\
 &= \frac{2}{(\sqrt{10} + \sqrt{10})(\sqrt{7+2\sqrt{10}} + (\sqrt{7+2\sqrt{10}}))} \\
 &= \frac{2}{(2\sqrt{10})(2\sqrt{7+2\sqrt{10}})} \\
 &= \frac{1}{(2\sqrt{10})(\sqrt{7+2\sqrt{10}})} \\
 &= \frac{1}{(2\sqrt{10})(\sqrt{5} + \sqrt{2})} \\
 &= \frac{(\sqrt{5} - \sqrt{2})}{(6\sqrt{10})}
 \end{aligned}$$

Q33

$$\begin{aligned}
 & \lim_{x \rightarrow \sqrt{6}} \frac{\sqrt{5+2x} - (\sqrt{3} + \sqrt{2})}{x^2 - 6} \\
 &= \lim_{x \rightarrow \sqrt{6}} \frac{\sqrt{5+2x} - \left(\sqrt{(\sqrt{3} + \sqrt{2})^2} \right)}{x^2 - 6} \\
 &= \lim_{x \rightarrow \sqrt{6}} \frac{\sqrt{5+2x} - \left(\sqrt{5+2\sqrt{6}} \right)}{x^2 - 6} \\
 &= \lim_{x \rightarrow \sqrt{6}} \frac{\sqrt{5+2x} - \left(\sqrt{5+2\sqrt{6}} \right)}{x^2 - 6} \times \frac{\sqrt{5+2x} + \left(\sqrt{5+2\sqrt{6}} \right)}{\sqrt{5+2x} + \left(\sqrt{5+2\sqrt{6}} \right)} \\
 &= \lim_{x \rightarrow \sqrt{6}} \frac{5+2x - 5 - 2\sqrt{6}}{(x^2 - 6) \left(\sqrt{5+2x} + \left(\sqrt{5+2\sqrt{6}} \right) \right)} \\
 &= \lim_{x \rightarrow \sqrt{6}} \frac{2(x - \sqrt{6})}{(x^2 - 6) \left(\sqrt{5+2x} + \left(\sqrt{5+2\sqrt{6}} \right) \right)} \\
 &= \lim_{x \rightarrow \sqrt{6}} \frac{2}{(x + \sqrt{6}) \left(\sqrt{5+2x} + \left(\sqrt{5+2\sqrt{6}} \right) \right)} \\
 &= \frac{2}{(\sqrt{6} + \sqrt{6}) \left(\sqrt{5+2\sqrt{6}} + \left(\sqrt{5+2\sqrt{6}} \right) \right)} \\
 &= \frac{2}{(2\sqrt{6}) (2\sqrt{5+2\sqrt{6}})} \\
 &= \frac{1}{(2\sqrt{6}) (\sqrt{5+2\sqrt{6}})} \\
 &= \frac{1}{(2\sqrt{6}) (\sqrt{3} + \sqrt{2})} \\
 &= \frac{(\sqrt{3} - \sqrt{2})}{(2\sqrt{6})}
 \end{aligned}$$

Q34

$$\begin{aligned}
 & \lim_{x \rightarrow \sqrt{2}} \frac{\sqrt{3+2x} - (\sqrt{2}+1)}{x^2 - 2} \\
 &= \lim_{x \rightarrow \sqrt{2}} \frac{\sqrt{3+2x} - \left(\sqrt{(\sqrt{2}+1)^2} \right)}{x^2 - 2} \\
 &= \lim_{x \rightarrow \sqrt{2}} \frac{\sqrt{3+2x} - (\sqrt{3+2\sqrt{2}})}{x^2 - 2} \\
 &= \lim_{x \rightarrow \sqrt{2}} \frac{\sqrt{3+2x} - (\sqrt{3+2\sqrt{2}})}{x^2 - 2} \times \frac{\sqrt{3+2x} + (\sqrt{3+2\sqrt{2}})}{\sqrt{3+2x} + (\sqrt{3+2\sqrt{2}})} \\
 &= \lim_{x \rightarrow \sqrt{2}} \frac{3+2x - 3 - 2\sqrt{2}}{(x^2 - 2)(\sqrt{3+2x} + (\sqrt{3+2\sqrt{2}}))} \\
 &= \lim_{x \rightarrow \sqrt{2}} \frac{2(x - \sqrt{2})}{(x^2 - 2)(\sqrt{3+2x} + (\sqrt{3+2\sqrt{2}}))} \\
 &= \lim_{x \rightarrow \sqrt{2}} \frac{2}{(x + \sqrt{2})(\sqrt{3+2x} + (\sqrt{3+2\sqrt{2}}))} \\
 &= \frac{2}{(\sqrt{2} + \sqrt{2})(\sqrt{3+2\sqrt{2}} + (\sqrt{3+2\sqrt{2}}))} \\
 &= \frac{2}{(2\sqrt{2})(2\sqrt{3+2\sqrt{2}})} \\
 &= \frac{1}{(2\sqrt{2})(\sqrt{3+2\sqrt{2}})} \\
 &= \frac{1}{(2\sqrt{2})(\sqrt{2}+1)} \\
 &= \frac{(\sqrt{2}-1)}{(2\sqrt{2})}
 \end{aligned}$$

Ex 29.5

Q1

$$\begin{aligned} & \lim_{x \rightarrow a} \frac{(x+2)^{\frac{5}{2}} - (a+2)^{\frac{5}{2}}}{x - a} \\ &= \lim_{x \rightarrow a} \frac{(x+2)^{\frac{5}{2}} - (a+2)^{\frac{5}{2}}}{(x+2) - (a+2)} \\ &= \lim_{x \rightarrow b} \frac{y^{\frac{5}{2}} - b^{\frac{5}{2}}}{y - b}, \text{ where } x+2 = y \text{ and } a+2 = b \\ &= \frac{5}{2} b^{\frac{5}{2}-1} \quad \left[\text{Using formula } \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = n a^{n-1} \right] \\ &= \frac{5}{2} (a+2)^{\frac{5}{2}-1} \\ &= \frac{5}{2} (a+2)^{\frac{3}{2}} \end{aligned}$$

Q2

$$\begin{aligned} & \lim_{x \rightarrow a} \frac{(x+2)^{\frac{3}{2}} - (a+2)^{\frac{3}{2}}}{x - a} \\ &= \lim_{x \rightarrow a} \frac{(x+2)^{\frac{3}{2}} - (a+2)^{\frac{3}{2}}}{(x+2) - (a+2)} \end{aligned}$$

Let $x+2 = y$, $a+2 = b$

$$\begin{aligned} & \Rightarrow \lim_{(x+2) \rightarrow (a+2)} \frac{(y)^{\frac{3}{2}} - (b)^{\frac{3}{2}}}{(y) - (b)} \quad \left[\text{Using formula } \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = n a^{n-1} \right] \\ &= \frac{3}{2} (b)^{\frac{3}{2}-1} \\ &= \frac{3}{2} (a+2)^{\frac{3}{2}-1} \\ &= \frac{3}{2} (a+2)^{\frac{1}{2}} \end{aligned}$$

Q3

$$\lim_{x \rightarrow 0} \frac{(1+x)^6 - 1}{(1+x)^2 - 1}$$

$$= \lim_{x \rightarrow 0} \frac{(1+x)^6 - 1^6}{(1+x)^2 - 1^2} \\ = \lim_{x \rightarrow 0} \frac{1+x-1}{1+x-1}$$

$\Rightarrow$ Let $1+x = y$, as $x \rightarrow 0$, $y \rightarrow 1$

$$= \lim_{y \rightarrow 1} \frac{y^6 - 1^6}{y - 1} \\ = \lim_{y \rightarrow 1} \frac{y^2 - 1}{y - 1}$$

$$= \frac{6(1)^{6-1}}{2(1)^{2-1}} \quad \left[\text{Using formula } \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1} \right] \\ = \frac{6}{2} \\ = 3$$

Q4

$$\lim_{x \rightarrow a} \frac{x^{\frac{2}{7}} - a^{\frac{2}{7}}}{x - a}$$

Applying formula $\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1}$, here, $n = \frac{2}{7}$

$$\Rightarrow \lim_{x \rightarrow a} \frac{x^{\frac{2}{7}} - a^{\frac{2}{7}}}{x - a} = \frac{2}{7} (a)^{\frac{2}{7}-1} \\ = \frac{2}{7} a^{-\frac{5}{7}}$$

Q5

$$\begin{aligned} & \lim_{x \rightarrow a} \frac{x^{\frac{5}{7}} - a^{\frac{5}{7}}}{x^{\frac{2}{7}} - a^{\frac{2}{7}}} \\ &= \lim_{x \rightarrow a} \frac{\frac{x^{\frac{5}{7}} - a^{\frac{5}{7}}}{x - a}}{\frac{x^{\frac{2}{7}} - a^{\frac{2}{7}}}{x - a}} \\ &= \lim_{x \rightarrow a} \frac{x^{\frac{5}{7}} - a^{\frac{5}{7}}}{x^{\frac{2}{7}} - a^{\frac{2}{7}}} \end{aligned}$$

[Dividing numerator and denominator by $x - a$]

Applying formula $\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1}$

Here, $n = \frac{5}{7}$ is numerator and applying $\lim_{x \rightarrow a} \frac{x^m - a^m}{x - a} = ma^{m-1}$ in denominator, where $m = \frac{2}{7}$

$$\begin{aligned} \Rightarrow \quad & \frac{\lim_{x \rightarrow a} \frac{x^{\frac{5}{7}} - a^{\frac{5}{7}}}{x - a}}{\lim_{x \rightarrow a} \frac{x^{\frac{2}{7}} - a^{\frac{2}{7}}}{x - a}} = \frac{\frac{5}{7}(a)^{\frac{5}{7}-1}}{\frac{2}{7}(a)^{\frac{2}{7}-1}} \\ &= \frac{\frac{5}{7}a^{-\frac{2}{7}}}{\frac{2}{7}a^{-\frac{5}{7}}} \\ &= \frac{5}{2}a^{-\frac{2}{7}+\frac{5}{7}} \\ &= \frac{5}{2}a^{\frac{3}{7}} \end{aligned}$$

Q6

$$\lim_{x \rightarrow -\frac{1}{2}} \frac{8x^3 + 1}{2x + 1}$$

$$= \frac{8}{2} \lim_{x \rightarrow -\frac{1}{2}} \frac{x^3 + \left(\frac{1}{2}\right)^3}{x + \frac{1}{2}}$$

$$= 4 \lim_{x \rightarrow -\frac{1}{2}} \frac{x^3 - \left(-\frac{1}{2}\right)^3}{x - \left(-\frac{1}{2}\right)}$$

Applying formula $\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1}$

Here, $n = 3$, $a = -\frac{1}{2}$

$$\begin{aligned} \Rightarrow 4 \lim_{x \rightarrow -\frac{1}{2}} \frac{x^3 - \left(-\frac{1}{2}\right)^3}{x - \left(-\frac{1}{2}\right)} &= 4 \times 3 \left(-\frac{1}{2}\right)^{3-1} \\ &= 4 \times 3 \times \frac{1}{4} \\ &= 3 \end{aligned}$$

Q7

$$\begin{aligned} & \lim_{x \rightarrow 27} \frac{\left(x^{\frac{1}{3}} + 3\right)\left(x^{\frac{1}{3}} - 3\right)}{x - 27} \\ &= \lim_{x \rightarrow 27} \frac{\left(x^{\frac{2}{3}} - 9\right)}{x - 27} \\ &= \lim_{x \rightarrow 27} \frac{x^{\frac{2}{3}} - 27^{\frac{2}{3}}}{x - 27} \end{aligned}$$

Applying formula $\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = n a^{n-1}$

$$\begin{aligned} &= \frac{2}{3} (27)^{\frac{2}{3}-1} \\ &= \frac{2}{3} (27)^{-\frac{1}{3}} \\ &= \frac{2}{3} \times \frac{1}{(27)^{\frac{1}{3}}} \\ &= \frac{2}{3} \times \frac{1}{3} \\ &= \frac{2}{9} \end{aligned}$$

Q8

$$\begin{aligned} & \lim_{x \rightarrow 4} \frac{x^3 - 64}{x^2 - 16} \\ &= \lim_{x \rightarrow 4} \frac{x^3 - 4^3}{x^2 - 4^2} \\ &= \lim_{x \rightarrow 4} \frac{\frac{x^3 - 4^3}{x - 4}}{\frac{x^2 - 4^2}{x - 4}} \\ &= \frac{\lim_{x \rightarrow 4} \frac{x^3 - 4^3}{x - 4}}{\lim_{x \rightarrow 4} \frac{x^2 - 4^2}{x - 4}} \end{aligned}$$

[Dividing numerator and denominator by $x - 4$]

Applying $\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1}$ in numerator and $\lim_{x \rightarrow a} \frac{x^m - a^m}{x - a} = ma^{m-1}$ in denominator

$$\Rightarrow n = 3, m = 2$$

$$\begin{aligned} \Rightarrow \frac{\lim_{x \rightarrow 4} \frac{x^3 - 4^3}{x - 4}}{\lim_{x \rightarrow 4} \frac{x^2 - 4^2}{x - 4}} &= \frac{3(4)^{3-1}}{2(4)^{2-1}} = \frac{3(4)^2}{2(4)} \\ &= 6 \end{aligned}$$

Q9

$$\begin{aligned} & \lim_{x \rightarrow 1} \frac{x^{15} - 1}{x^{10} - 1} \\ &= \lim_{x \rightarrow 1} \frac{x^{15} - 1^{15}}{x^{10} - 1^{10}} \quad \text{[Dividing numerator and denominator by } (x-1)\text{]} \\ &= \lim_{x \rightarrow 1} \frac{x-1}{x-1} \cdot \frac{x^{15} - 1^{15}}{x^{10} - 1^{10}} \\ &= \lim_{x \rightarrow 1} \frac{x^{15} - 1^{15}}{x^{10} - 1^{10}} \end{aligned}$$

Applying formula $\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1}$ in numerator and $\lim_{x \rightarrow a} \frac{x^m - a^m}{x - a} = ma^{m-1}$ in denominator

Here, $n = 15$, $m = 10$

$$\begin{aligned} \Rightarrow \quad & \frac{\lim_{x \rightarrow 1} \frac{x^{15} - 1^{15}}{x - 1}}{\lim_{x \rightarrow 1} \frac{x^{10} - 1^{10}}{x - 1}} = \frac{15(1)^{15-1}}{10(1)^{10-1}} \\ &= \frac{15}{10} \\ &= \frac{3}{2} \end{aligned}$$

Q10

$$\begin{aligned} & \lim_{x \rightarrow -1} \frac{x^3 + 1}{x + 1} \\ &= \lim_{x \rightarrow -1} \frac{x^3 - (-1)^3}{x - (-1)} \end{aligned}$$

Applying formula $\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1}$

Here, $n = 3$, $a = -1$

$$\begin{aligned} \Rightarrow \quad & \lim_{x \rightarrow -1} \frac{x^3 - (-1)^3}{x - (-1)} = na^{n-1} \\ &= 3(-1)^{3-1} \\ &= 3(-1)^2 \\ &= 3 \end{aligned}$$

Q11

$$\begin{aligned} & \lim_{x \rightarrow a} \frac{x^{\frac{2}{3}} - a^{\frac{2}{3}}}{x^{\frac{3}{4}} - a^{\frac{3}{4}}} \\ &= \lim_{x \rightarrow a} \frac{\frac{x^{\frac{2}{3}} - a^{\frac{2}{3}}}{x - a}}{\frac{x^{\frac{3}{4}} - a^{\frac{3}{4}}}{x - a}} \\ &= \lim_{x \rightarrow a} \frac{x^{\frac{2}{3}} - a^{\frac{2}{3}}}{x - a} \cdot \lim_{x \rightarrow a} \frac{x - a}{x^{\frac{3}{4}} - a^{\frac{3}{4}}} \end{aligned}$$

[Dividing numerator and denominator by $x - a$]

Applying the formula $\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1}$ in numerator and $\lim_{x \rightarrow a} \frac{x^m - a^m}{x - a} = ma^{m-1}$ in denominator respectively

Here, $n = \frac{2}{3}$, $m = \frac{3}{4}$

$$\begin{aligned} \Rightarrow & \frac{\lim_{x \rightarrow a} \frac{x^{\frac{2}{3}} - a^{\frac{2}{3}}}{x - a}}{\lim_{x \rightarrow a} \frac{x^{\frac{3}{4}} - a^{\frac{3}{4}}}{x - a}} = \frac{\frac{2}{3}(a)^{\frac{2}{3}-1}}{\frac{3}{4}(a)^{\frac{3}{4}-1}} \\ &= \frac{8}{9} a^{\frac{-1}{3} + \frac{1}{4}} \\ &= \frac{8}{9} a^{\frac{-1}{12}} \end{aligned}$$

Q12

$$\lim_{x \rightarrow 3} \frac{x^n - 3^n}{x - 3} = 108$$

$$\text{LHS} = \lim_{x \rightarrow 3} \frac{x^n - 3^n}{x - 3}$$

$$\text{Applying the formula } \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = n a^{n-1}$$

$$\text{Here, } n = n, a = 3$$

$$\Rightarrow \lim_{x \rightarrow 3} \frac{x^n - 3^n}{x - 3} = n (3)^{n-1}$$

$$\text{It is given that } n (3)^{n-1} = 108$$

$$\begin{aligned} \Rightarrow n (3)^{n-1} &= 2 \times 2 \times 3 \times 3 \times 3 \\ &= (2)^2 \times (3)^3 \\ &= 4 (3)^{4-1} \end{aligned}$$

$$\Rightarrow n = 4$$

Q13

$$\text{If } \lim_{x \rightarrow a} \frac{x^9 - a^9}{x - a} = 9 \quad \text{---(i)}$$

$$\text{LHS} = \lim_{x \rightarrow a} \frac{x^9 - a^9}{x - a}$$

$$\begin{aligned} &= 9 (a)^{9-1} \\ &= 9a^8 \end{aligned}$$

$$\text{It is given that } 9a^8 = 9 \quad [\text{From (i)}]$$

$$\Rightarrow a^8 = \frac{9}{9} = 1$$

$$\begin{aligned} \Rightarrow a^4 &= 1 \\ a^2 &= 1 \\ a &= \pm 1 \end{aligned}$$

$$\Rightarrow a = 1 \text{ and } a = -1$$

Q14

$$\text{If } \lim_{x \rightarrow a} \frac{x^5 - a^5}{x - a} = 405 \quad \text{---(i)}$$

$$\begin{aligned} \text{LHS} &= \lim_{x \rightarrow a} \frac{x^5 - a^5}{x - a} \\ &= 5(a)^{5-1} \\ &= 5a^4 \end{aligned}$$

$$\text{It is given that } 5a^4 = 405$$

$$\begin{aligned} \Rightarrow \quad 5a^4 &= 405 \\ a^4 &= \frac{405}{5} = 81 \\ a^4 &= (3)^4, \quad a^2 = 9 \\ a &= \pm 3 \end{aligned}$$

$$\Rightarrow \quad a = 3 \text{ and } a = -3$$

Q15

$$\text{If } \lim_{x \rightarrow 5} \frac{x^9 - a^9}{x - a} = \lim_{x \rightarrow 5} (4 + x) \quad \text{---(i)}$$

$$\begin{aligned} \text{LHS} &= \lim_{x \rightarrow a} \frac{x^9 - a^9}{x - a} \\ &= 9(a)^{9-1} = 9a^8 \end{aligned} \quad \text{---(ii)}$$

$$\begin{aligned} \text{RHS} &= \lim_{x \rightarrow 5} (4 + x) \\ &= 4 + 5 = 9 \end{aligned} \quad \text{---(iii)}$$

Substituting (ii) and (iii) in (i),

$$\begin{aligned} 9a^8 &= 9 \\ a^8 &= 1 \\ \Rightarrow \quad a^2 &= 1 \\ \Rightarrow \quad a &= 1, \quad a = -1 \end{aligned}$$

Q16

$$\text{If } \lim_{x \rightarrow 2} \frac{x^3 - a^3}{x - a} = \lim_{x \rightarrow 1} \frac{x^4 - 1}{x - 1} \quad \text{--- (i)}$$

$$\begin{aligned} \text{LHS} &= \lim_{x \rightarrow 2} \frac{x^3 - a^3}{x - a} \\ &= 3(a)^{3-1} \\ &= 3a^2 \end{aligned} \quad \text{--- (ii)}$$

$$\begin{aligned} \text{RHS} &= \lim_{x \rightarrow 2} \frac{x^4 - 1}{x - 1} \\ &= \lim_{x \rightarrow 2} \frac{x^4 - 1}{x - 1} \\ &= 4(1)^{4-1} \\ &= 4 \end{aligned} \quad \text{--- (iii)}$$

Substituting (ii) and (iii) in (i),

$$3a^2 = 4$$

$$a^2 = \frac{4}{3}$$

$$a = \pm \frac{2}{\sqrt{3}}$$

Ex 29.6

Q1

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{(3x-1)(4x-2)}{(x+8)(x-1)} & \quad \left[\text{Expression is } \frac{\infty}{\infty} \right] \\ &= \lim_{x \rightarrow \infty} \frac{(12x^2 - 10x + 2)}{(x^2 + 9x - 8)} \\ &= \lim_{x \rightarrow \infty} \left(\frac{12 - \frac{10}{x} + \frac{2}{x^2}}{1 + \frac{9}{x} - \frac{8}{x^2}} \right) \\ &= \frac{12 - 0 + 0}{1 + 0 - 0} \\ &= 12 \end{aligned}$$

Q2

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{3x^3 - 4x^2 + 6x - 1}{2x^3 + x^2 - 5x + 7} \\ &= \lim_{x \rightarrow \infty} \frac{3 - \frac{4}{x} + \frac{6}{x^2} - \frac{1}{x^3}}{2 + \frac{1}{x} - \frac{5}{x^2} + \frac{7}{x^3}} \\ &= \frac{3 - 0 + 0 - 0}{2 + 0 - 0 + 0} \\ &= \frac{3}{2} \end{aligned}$$

Q3

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{5x^3 - 6}{\sqrt{9 + 4x^6}} \\ &= \lim_{x \rightarrow \infty} \frac{5 - \frac{6}{x^3}}{\sqrt{\frac{9}{x^6} + \frac{4x^6}{x^6}}} \\ &= \lim_{x \rightarrow \infty} \frac{\left(5 - \frac{6}{x^3}\right)}{\sqrt{\frac{9}{x^6} + 4}} \\ &= \frac{5}{\sqrt{4}} = \frac{5}{2} \end{aligned}$$

Q4

$$\begin{aligned} & \lim_{x \rightarrow \infty} \sqrt{x^2 + cx} - x \\ &= \lim_{x \rightarrow \infty} \left(\left(\sqrt{x^2 + cx} - x \right) \frac{\left(\sqrt{x^2 + cx} + x \right)}{\left(\sqrt{x^2 + cx} + x \right)} \right) \\ &= \lim_{x \rightarrow \infty} \frac{\left(x^2 + cx - x^2 \right)}{\sqrt{x^2 + cx} + x} \\ &= \lim_{x \rightarrow \infty} \frac{cx}{\sqrt{x^2 + cx} + x} \quad \left[\frac{\infty}{\infty} \text{ form} \right] \\ &= \lim_{x \rightarrow \infty} \frac{c}{\sqrt{1 + \frac{c}{x}} + 1} \\ &= \frac{c}{1+1} = \frac{c}{2} \end{aligned}$$

Q5

$$\begin{aligned} & \lim_{x \rightarrow \infty} \sqrt{x+1} - \sqrt{x} \\ &= \lim_{x \rightarrow \infty} \frac{\left(\sqrt{x+1} - \sqrt{x} \right) \left(\sqrt{x+1} + \sqrt{x} \right)}{\left(\sqrt{x+1} + \sqrt{x} \right)} \\ &= \lim_{x \rightarrow \infty} \frac{(x+1-x)}{\sqrt{x+1} + \sqrt{x}} \\ &= \lim_{x \rightarrow \infty} \left(\frac{1}{\sqrt{x+1} + \sqrt{x}} \right) \\ &= \frac{1}{\infty} \\ &= 0 \end{aligned}$$

Q6

$$\begin{aligned}\lim_{x \rightarrow \infty} \sqrt{x^2 + 7x} - x \\&= \lim_{x \rightarrow \infty} \left(\frac{(\sqrt{x^2 + 7x} - x)(\sqrt{x^2 + 7x} + x)}{\sqrt{x^2 + 7x} + x} \right) \\&= \lim_{x \rightarrow \infty} \left(\frac{(x^2 + 7x) - x^2}{\sqrt{x^2 + 7x} + x} \right) \\&= \lim_{x \rightarrow \infty} \frac{7x}{\sqrt{x^2 + 7x} + x} \\&= \lim_{x \rightarrow \infty} \frac{7}{\sqrt{\frac{x^2}{x^2} + \frac{7x}{x^2}} + 1} \\&= \lim_{x \rightarrow \infty} \frac{7}{\sqrt{1 + \frac{7}{x}} + 1} \\&= \frac{7}{2}\end{aligned}$$

Q7

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{x}{\sqrt{4x^2 + 1} - 1} \\&= \lim_{x \rightarrow \infty} \frac{1}{\sqrt{4 + \frac{1}{x^2}} - \frac{1}{x}} \\&= \frac{1}{\sqrt{4} - 0} \\&= \frac{1}{2}\end{aligned}$$

Q8

$$\lim_{n \rightarrow \infty} \frac{n^2}{1+2+3+\dots+n}$$

$$= \lim_{n \rightarrow \infty} \frac{n^2}{\frac{1}{2}n(n+1)}$$

$$= \lim_{n \rightarrow \infty} \frac{2n^2}{n^2+n}$$

$$= 2 \lim_{n \rightarrow \infty} \frac{n^2}{n^2+n}$$

$$= 2 \lim_{n \rightarrow \infty} \frac{n^2}{n^2 \left(1 + \frac{1}{n}\right)}$$

$$= 2 \times \frac{1}{1+0}$$

$$= 2$$

$$\left[\because 1+2+3+\dots+n = \frac{n(n+1)}{2} \right]$$

Q9

$$\lim_{x \rightarrow \infty} \frac{\frac{3}{x} + \frac{4}{x^2}}{\frac{5}{x} + \frac{6}{x^2}}$$

$$= \lim_{x \rightarrow \infty} \frac{\frac{1}{x} \left(3 + \frac{4}{x}\right)}{\frac{1}{x} \left(5 + \frac{6}{x}\right)}$$

$$= \lim_{x \rightarrow \infty} \frac{(3+0)}{(5+0)} = \frac{3}{5}$$

$$\left[\frac{0}{0} \text{ form} \right]$$

Q10

$$\lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + a^2} - \sqrt{x^2 + b^2}}{\sqrt{x^2 + c^2} - \sqrt{x^2 + d^2}} \quad \left[\frac{\infty}{\infty} \text{ form} \right]$$

$$\begin{aligned} & \lim_{x \rightarrow \infty} \frac{\left(\sqrt{x^2 + a^2} - \sqrt{x^2 + b^2} \right) \left(\sqrt{x^2 + a^2} + \sqrt{x^2 + b^2} \right)}{\left(\sqrt{x^2 + c^2} - \sqrt{x^2 + d^2} \right) \left(\sqrt{x^2 + c^2} + \sqrt{x^2 + d^2} \right)} \\ &= \lim_{x \rightarrow \infty} \frac{\left((x^2 + a^2) - (x^2 + b^2) \right)}{\left(\sqrt{x^2 + c^2} - \sqrt{x^2 + d^2} \right) \left(\sqrt{x^2 + a^2} + \sqrt{x^2 + b^2} \right)} \\ &= \lim_{x \rightarrow \infty} \frac{(a^2 - b^2)}{\left(\sqrt{x^2 + c^2} - \sqrt{x^2 + d^2} \right) \left(\sqrt{x^2 + a^2} + \sqrt{x^2 + b^2} \right)} \\ &= \lim_{x \rightarrow \infty} \frac{(a^2 - b^2) \left(\sqrt{x^2 + c^2} + \sqrt{x^2 + d^2} \right)}{\left(\sqrt{x^2 + c^2} - \sqrt{x^2 + d^2} \right) \left(\sqrt{x^2 + c^2} + \sqrt{x^2 + d^2} \right) \left(\sqrt{x^2 + a^2} + \sqrt{x^2 + b^2} \right)} \\ &= \lim_{x \rightarrow \infty} \frac{(a^2 - b^2) \left(\sqrt{x^2 + c^2} + \sqrt{x^2 + d^2} \right)}{(x^2 + c^2 - x^2 - d^2) \left(\sqrt{x^2 + a^2} + \sqrt{x^2 + b^2} \right)} \\ &= \lim_{x \rightarrow \infty} \frac{(a^2 - b^2) \left(\sqrt{x^2 + c^2} + \sqrt{x^2 + d^2} \right)}{(c^2 - d^2) \left(\sqrt{x^2 + a^2} + \sqrt{x^2 + b^2} \right)} \\ &= \lim_{x \rightarrow \infty} \frac{(a^2 - b^2) \left(x \sqrt{1 + \frac{c^2}{x^2}} + x \sqrt{1 + \frac{d^2}{x^2}} \right)}{(c^2 - d^2) \left(x \sqrt{1 + \frac{a^2}{x^2}} + x \sqrt{1 + \frac{b^2}{x^2}} \right)} \\ &= \lim_{x \rightarrow \infty} \frac{(a^2 - b^2) \left(\sqrt{1 + \frac{c^2}{x^2}} + \sqrt{1 + \frac{d^2}{x^2}} \right)}{(c^2 - d^2) \left(\sqrt{1 + \frac{a^2}{x^2}} + \sqrt{1 + \frac{b^2}{x^2}} \right)} \\ &= \frac{(a^2 - b^2) \left(\sqrt{1 + 0} + \sqrt{1 + 0} \right)}{(c^2 - d^2) \left(\sqrt{1 + 0} + \sqrt{1 + 0} \right)} \\ &= \frac{(a^2 - b^2) (1 + 1)}{(c^2 - d^2) (1 + 1)} \\ &= \frac{(a^2 - b^2) (2)}{(c^2 - d^2) (2)} = \frac{a^2 - b^2}{c^2 - d^2} \end{aligned}$$

Q11

$$\lim_{n \rightarrow \infty} \frac{(n+2)! + (n+1)!}{(n+2)! - (n+1)!}$$

We know that $(n+2)! = (n+2)(n+1)!$

$$\begin{aligned} \Rightarrow \quad & \lim_{n \rightarrow \infty} \frac{(n+2)(n+1)! + (n+1)!}{(n+2)(n+1)! - (n+1)!} \\ &= \lim_{n \rightarrow \infty} \frac{(n+1)![(n+2)+1]}{(n+1)![(n+2)-1]} \\ &= \lim_{n \rightarrow \infty} \frac{n+3}{n+1} \\ &= \lim_{n \rightarrow \infty} \frac{1 + \frac{3}{n}}{1 + \frac{1}{n}} \\ &= \frac{1+0}{1+0} = 1 \\ &= 1 \end{aligned}$$

$\left[\frac{\infty}{\infty} \text{ form} \right]$

Q12

$$\begin{aligned} & \lim_{x \rightarrow \infty} x \left[\sqrt{x^2+1} - \sqrt{x^2-1} \right] \\ &= \lim_{x \rightarrow \infty} x \left[\sqrt{x^2+1} - \sqrt{x^2-1} \right] \times \frac{\left(\sqrt{x^2+1} + \sqrt{x^2-1} \right)}{\left(\sqrt{x^2+1} + \sqrt{x^2-1} \right)} \\ &= \lim_{x \rightarrow \infty} x \frac{x \left(x^2+1 - x^2+1 \right)}{\sqrt{x^2+1} + \sqrt{x^2-1}} \\ &= \lim_{x \rightarrow \infty} \frac{x(2)}{x \left(\sqrt{1+\frac{1}{x^2}} + \sqrt{1-\frac{1}{x^2}} \right)} \\ &= \lim_{x \rightarrow \infty} \frac{2}{\sqrt{1+\frac{1}{x^2}} + \sqrt{1-\frac{1}{x^2}}} \\ &= \frac{2}{2} = 1 \end{aligned}$$

Q13

$$\begin{aligned}
 & \lim_{x \rightarrow \infty} \left[\sqrt{x+1} - \sqrt{x} \right] \sqrt{x+2} \\
 &= \lim_{x \rightarrow \infty} \left[\sqrt{x+1} - \sqrt{x} \right] \frac{\left[\sqrt{x+1} + \sqrt{x} \right]}{\left[\sqrt{x+1} + \sqrt{x} \right]} \times \frac{\sqrt{x+2} \times \sqrt{x+2}}{\sqrt{x+2}} \\
 &= \lim_{x \rightarrow \infty} \frac{(x+1-x)}{\sqrt{x+1} + \sqrt{x}} \times \frac{(x+2)}{\sqrt{x+2}} \\
 &= \lim_{x \rightarrow \infty} \frac{1(x+2)}{\left(\sqrt{x+1} + \sqrt{x} \right) \left(\sqrt{x+2} \right)} \\
 &= \lim_{x \rightarrow \infty} \frac{x \left(1 + \frac{2}{x} \right)}{\sqrt{x} \left(\sqrt{1 + \frac{1}{x}} + \sqrt{1} \right) \left(\sqrt{1 + \frac{2}{x}} \right) \sqrt{x}} \\
 &= \lim_{x \rightarrow \infty} \frac{\left(1 + \frac{2}{x} \right)}{\left(\sqrt{1 + \frac{1}{x}} + \sqrt{1} \right) \left(\sqrt{1 + \frac{2}{x}} \right)} \\
 &= \lim_{x \rightarrow \infty} \frac{(1+0)}{(1+1) \times 1} = \frac{1}{2}
 \end{aligned}$$

Q14

$$\lim_{n \rightarrow \infty} \frac{1^2 + 2^2 + 3^2 + \dots + n^2}{n^3}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{6} \frac{n(n+1)(2n+1)}{n^3}$$

$$= \frac{1}{6} \lim_{n \rightarrow \infty} \frac{(n^2 + n)(2n+1)}{n^3}$$

$$= \frac{1}{6} \lim_{n \rightarrow \infty} \frac{(2n^3 + n^2 + 2n^2 + n)}{n^3}$$

$$= \frac{1}{6} \lim_{n \rightarrow \infty} \frac{(2n^3 + 3n^2 + n)}{n^3}$$

$$= \frac{1}{6} \lim_{n \rightarrow \infty} \frac{\left(2 + \frac{3}{n} + \frac{1}{n^2} \right)}{1}$$

$$= \frac{1}{6} \frac{(2+0+0)}{1} = \frac{1}{3}$$

$$\left| 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{1}{6} n(n+1)(2n+1) \right|$$

$$\left[\frac{1}{6} n^3 \text{ term} \right]$$

Q15

$$\lim_{n \rightarrow \infty} \left(\frac{1}{n^2} + \frac{2}{n^2} + \frac{3}{n^2} + \dots + \frac{n-1}{n^2} \right)$$

$$\lim_{n \rightarrow \infty} \left(\frac{1+2+3+\dots+(n-1)}{n^2} \right)$$

$$= \lim_{n \rightarrow \infty} \frac{(n-1)(n)}{2 \times n^2}$$

$$= \lim_{n \rightarrow \infty} \frac{n^2 - n}{2n^2}$$

$$= \lim_{n \rightarrow \infty} \frac{1 - \frac{1}{n}}{2}$$

$$= \frac{1 - 0}{2} = \frac{1}{2}$$

$$= \frac{1}{2}$$

$$\left[1+2+3+\dots+(n-1) = \frac{(n-1)(n)}{2} \right]$$

$$\left[\frac{\infty}{\infty} \text{ form} \right]$$

Q16

$$\lim_{n \rightarrow \infty} \frac{1^3 + 2^3 + 3^3 + \dots + n^3}{n^4}$$

$$= \lim_{n \rightarrow \infty} \frac{\left[\frac{1}{2} n(n+1) \right]^2}{n^4}$$

$$= \lim_{n \rightarrow \infty} \frac{\frac{1}{4} n^2 (n+1)^2}{n^4}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{4} \frac{(n^2 + 1 + 2n)}{n^2}$$

$$= \frac{1}{4} \lim_{n \rightarrow \infty} \frac{\left(1 + \frac{1}{n^2} + \frac{2}{n} \right)}{1}$$

$$= \frac{1}{4}$$

$$\left[1^3 + 2^3 + 3^3 + \dots + n^3 = \left(\frac{1}{2} n(n+1) \right)^2 \right]$$

$$[\text{Multiplying the term}] \left[\frac{\infty}{\infty} \text{ form} \right]$$

Q17

$$\lim_{n \rightarrow \infty} \frac{1^3 + 2^3 + \dots + n^3}{(n-1)^4}$$

$$= \lim_{n \rightarrow \infty} \frac{\left[\frac{1}{2} n(n+1) \right]^2}{(n-1)^4}$$

$$= \lim_{n \rightarrow \infty} \left(\frac{\frac{1}{4} n^2 (n^2 + 1 + 2n)}{(n-1)^4} \right)$$

$$= \frac{1}{4} \lim_{n \rightarrow \infty} \left(\frac{n^4 + n^2 + 2n^3}{(n-1)^2 (n-1)^2} \right)$$

$$= \frac{1}{4} \lim_{n \rightarrow \infty} \left(\frac{n^4 + n^2 + 2n^3}{n^4 + n^2 - 2n^3 + n^2 + 1 - 2n - 2n^3 - 2n + 4n^2} \right)$$

$$= \frac{1}{4} \lim_{n \rightarrow \infty} \frac{\left(1 + \frac{1}{n^2} + \frac{2}{n} \right)}{\left(1 + \frac{1}{n^2} - \frac{2}{n} + \frac{1}{n^2} + \frac{1}{n^4} - \frac{2}{n^3} - \frac{2}{n} - \frac{2}{n^3} + \frac{4}{n^2} \right)}$$

$$= \frac{1}{4} \left(\frac{1}{1} \right)$$

$$= \frac{1}{4}$$

$$\left[1^3 + 2^3 + 3^3 + \dots + n^3 = \left[\frac{1}{2} n(n+1) \right]^2 \right]$$

$$\left[\frac{\infty}{\infty} \text{ form} \right]$$

Q18

$$\lim_{n \rightarrow \infty} \sqrt{x} \left(\sqrt{x+1} - \sqrt{x} \right)$$

$$= \lim_{n \rightarrow \infty} \left(\sqrt{x^2 + x} - x \right)$$

$$= \lim_{n \rightarrow \infty} \left(\left(\sqrt{x^2 + x} - x \right) \times \frac{\left(\sqrt{x^2 + x} + x \right)}{\sqrt{x^2 + x} + x} \right)$$

$$= \lim_{n \rightarrow \infty} \left(\frac{\left(x^2 + x \right) - x^2}{\sqrt{x^2 + x} + x} \right)$$

$$= \lim_{n \rightarrow \infty} \left(\frac{x}{\sqrt{x^2 + x} + x} \right)$$

 $\left[\frac{\infty}{\infty} \text{ form} \right]$

$$= \lim_{n \rightarrow \infty} \left(\frac{1}{\sqrt{\frac{x^2}{x^2} + \frac{x}{x^2}} + 1} \right)$$

$$= \lim_{n \rightarrow \infty} \left(\frac{1}{\sqrt{1 + \frac{1}{x}} + 1} \right)$$

$$= \frac{1}{1+1}$$

$$= \frac{1}{2}$$

Q19

$$\lim_{n \rightarrow \infty} \left(\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots + \frac{1}{3^n} \right) \quad \text{---(i)}$$

This is G.P of common ratio $\frac{1}{3}$

$\therefore$ Sum of n terms of G.P with $a = \frac{1}{3}, r = \frac{1}{3}$

$$\begin{aligned} S_n &= a \left(\frac{1-r^n}{1-r} \right) \\ S_n &= \frac{1}{3} \left(\frac{1 - \left(\frac{1}{3} \right)^n}{1 - \frac{1}{3}} \right) \\ &= \frac{1}{3} \left(\frac{1 - \frac{1}{3^n}}{\frac{2}{3}} \right) \\ &= \frac{1}{3} \times \frac{3}{2} \left(1 - \frac{1}{3^n} \right) \\ S_n &= \frac{1}{2} \left(1 - \frac{1}{3^n} \right) \end{aligned}$$

Substituting value of S_n in (i), we get

$$\begin{aligned} \lim_{n \rightarrow \infty} S_n &= \lim_{n \rightarrow \infty} \frac{1}{2} \left(1 - \frac{1}{3^n} \right) \\ &= \frac{1}{2} \lim_{n \rightarrow \infty} \left(1 - \frac{1}{3^n} \right) \\ &= \frac{1}{2} (1 - 0) \\ &= \frac{1}{2} \end{aligned}$$

Q20

$$\lim_{x \rightarrow \infty} \frac{x^4 + 7x^3 + 46x + a}{x^4 + 6}$$

$\left[\frac{\infty}{\infty} \text{ form} \right]$

$$= \lim_{x \rightarrow \infty} \frac{1 + \frac{7}{x} + \frac{46}{x^3} + \frac{a}{x^4}}{1 + \frac{6}{x^4}}$$

$$= 1$$

Q21

$$f(x) = \frac{ax^2 + b}{x^2 + 1}$$

Also $\lim_{x \rightarrow 0} f(x) = 1$ ---(i) [given]

$$\Rightarrow \lim_{x \rightarrow 0} \frac{ax^2 + b}{x^2 + 1} = 1$$

$$\Rightarrow \frac{\lim_{x \rightarrow 0} ax^2 + b}{\lim_{x \rightarrow 0} x^2 + 1} = 1$$

$$\Rightarrow b = 1$$
 [from (i)]

Also, it is given that $\lim_{x \rightarrow \infty} f(x) = 1$

$$\therefore \lim_{x \rightarrow \infty} \frac{ax^2 + b}{x^2 + 1} = 1$$
 ---(ii)

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{ax^2 + 1}{x^2 + 1} = 1$$

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{a + \frac{1}{x^2}}{1 + \frac{1}{x^2}} = 1$$
 [$\frac{\infty}{\infty}$ form]

$$\Rightarrow a = 1$$

Thus, $f(x) = \frac{ax^2 + b}{x^2 + 1} = \frac{x^2 + 1}{x^2 + 1} = 1$ [from (ii)]

$$f(-2) = 1$$

$$f(2) = 1$$

$$f(-2) = 1 = f(2)$$

Hence, proved.

Q22

$$\begin{aligned}
 \text{RHS} &= \lim_{x \rightarrow 0} \left(\sqrt{x^2 + 1} - x \right) \\
 &= \lim_{x \rightarrow 0} \frac{\left(\sqrt{x^2 + 1} - x \right) \left(\sqrt{x^2 + 1} + x \right)}{\left(\sqrt{x^2 + 1} + x \right)} \\
 &= \lim_{x \rightarrow 0} \frac{x^2 + 1 - x^2}{x \sqrt{1 + \frac{1}{x^2}} + x} \\
 &= \lim_{x \rightarrow 0} \frac{1}{x \sqrt{1 + \frac{1}{x^2}} + x} \\
 &= \lim_{x \rightarrow 0} \frac{1}{x \left(\sqrt{1 + \frac{1}{x^2}} + 1 \right)} \\
 &= 0
 \end{aligned}$$

Also,

$$\begin{aligned}
 \text{LHS} &= \lim_{x \rightarrow 0} \left(\sqrt{x^2 + x + 1} - x \right) \\
 &= \lim_{x \rightarrow 0} \frac{\left(\sqrt{x^2 + x + 1} - x \right) \left(\sqrt{x^2 + x + 1} + x \right)}{\left(\sqrt{x^2 + x + 1} + x \right)} \\
 &= \lim_{x \rightarrow 0} \frac{x^2 + x + 1 - x^2}{\sqrt{x^2 + x + 1} + x} \\
 &= \lim_{x \rightarrow 0} \frac{x \left(1 + \frac{1}{x} \right)}{x \left(\sqrt{1 + \frac{1}{x} + \frac{1}{x^2}} + 1 \right)} \\
 &= \lim_{x \rightarrow 0} \frac{1 + \frac{1}{x}}{\sqrt{1 + \frac{1}{x} + \frac{1}{x^2}} + 1} \\
 &= \frac{1}{1 + 1} = \frac{1}{2}
 \end{aligned}$$

LHS $\neq$ RHS

$$\therefore \lim_{x \rightarrow 0} \left(\sqrt{x^2 + x + 1} - x \right) \text{ is not equal to } \lim_{x \rightarrow 0} \left(\sqrt{x^2 + 1} - x \right).$$

Q23

$$\lim_{x \rightarrow -\infty} (\sqrt{4x^2 - 7x} + 2x)$$

Substitute $y = -x$

$$= \lim_{y \rightarrow \infty} (\sqrt{4y^2 + 7y} - 2y)$$

$$= \lim_{y \rightarrow \infty} \frac{(\sqrt{4y^2 + 7y} - 2y)(\sqrt{4y^2 + 7y} + 2y)}{\sqrt{4y^2 + 7y} + 2y}$$

$$= \lim_{y \rightarrow \infty} \frac{(4y^2 + 7y - 4y^2)}{\sqrt{4y^2 + 7y} + 2y}$$

$$= \lim_{y \rightarrow \infty} \frac{(7y)}{\sqrt{4y^2 + 7y} + 2y}$$

$$= \lim_{y \rightarrow \infty} \frac{7}{\sqrt{4 + \frac{7}{y}} + 2}$$

$$= \frac{7}{2+2} = \frac{7}{4}$$

Q24

$$\begin{aligned}
 & \lim_{x \rightarrow -\infty} \left(\sqrt{x^2 - 8x} + x \right) \\
 &= \lim_{y \rightarrow \infty} \left(\sqrt{y^2 + 8y} - y \right), \text{ where } y = -x \text{ on rationalising} \\
 &= \lim_{y \rightarrow \infty} \frac{\left\{ \sqrt{y^2 + 8y} - y \right\} \left\{ \sqrt{y^2 + 8y} + y \right\}}{\left\{ \sqrt{y^2 + 8y} + y \right\}} \\
 &= \lim_{y \rightarrow \infty} \frac{y^2 + 8y - y^2}{\sqrt{y^2 + 8y} + y} \\
 &= \lim_{y \rightarrow \infty} \frac{8y}{y \sqrt{1 + \frac{8}{y}} + y} \quad \left[\frac{\infty}{\infty} \text{ form} \right] \\
 &= \lim_{y \rightarrow \infty} \frac{8}{\sqrt{1 + \frac{8}{y}} + 1} \\
 &= \frac{8}{1 + 1} = \frac{8}{2} \\
 &= 4
 \end{aligned}$$

Q25

$$\begin{aligned}
 & \lim_{n \rightarrow \infty} \frac{1^4 + 2^4 + 3^4 + \dots + n^4}{n^5} - \lim_{n \rightarrow \infty} \frac{1^3 + 2^3 + \dots + n^3}{n^5} \\
 &= \lim_{n \rightarrow \infty} \frac{\frac{n(1+n)(1+2n)(-1+3n+3n^2)}{30}}{n^5} - \lim_{n \rightarrow \infty} \frac{\left(\frac{n(n+1)}{2} \right)^2}{n^5} \\
 &= \lim_{n \rightarrow \infty} \frac{\left(\frac{1}{n} + 1 \right) \left(\frac{1}{n} + 2 \right) \left(-\frac{1}{n^2} + \frac{3}{n} + 3 \right)}{30} - \lim_{n \rightarrow \infty} \frac{1}{n^5} \left(\frac{n^2(n^2 + 2n + 1)}{4} \right) \\
 &= \lim_{n \rightarrow \infty} \frac{\left(\frac{1}{n} + 1 \right) \left(\frac{1}{n} + 2 \right) \left(-\frac{1}{n^2} + \frac{3}{n} + 3 \right)}{30} - \lim_{n \rightarrow \infty} \frac{\left(\frac{1}{n} + \frac{2}{n^2} + \frac{1}{n^3} \right)}{4} \\
 &= \frac{1 \times 2 \times 3}{30} - 0 \\
 &= \frac{1}{5}
 \end{aligned}$$

Q26

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{1.2 + 2.3 + 3.4 + \dots + n(n+1)}{n^3} &= \lim_{n \rightarrow \infty} \frac{\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}}{n^3} \\&= \lim_{n \rightarrow \infty} \frac{n(n+1) \left[\frac{(2n+1)+3}{6} \right]}{n^3} \\&= \lim_{n \rightarrow \infty} \frac{n(n+1)(2n+4)}{6n^3} \\&= \lim_{n \rightarrow \infty} \frac{\left(1 + \frac{1}{n}\right) \left(2 + \frac{4}{n}\right)}{6} \\&= \frac{1 \times 2}{6} \\&= \frac{1}{3}\end{aligned}$$

Ex 29.7

Q1

$$\begin{aligned}
 & \lim_{x \rightarrow 0} \frac{\sin 3x}{5x} \\
 &= \frac{1}{5} \lim_{x \rightarrow 0} \frac{\sin 3x}{x} \\
 &= \frac{1}{5} \lim_{x \rightarrow 0} \frac{\sin 3x}{3x} \times 3 \\
 &= \frac{3}{5} \lim_{3x \rightarrow 0} \frac{\sin 3x}{3x} \\
 &= \frac{3}{5} \times 1 \\
 &= \frac{3}{5}
 \end{aligned}$$

$$\left[\because \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \right]$$

Q2

$$\begin{aligned}
 & \lim_{x \rightarrow 0} \frac{\sin x^\circ}{x} \\
 &= \lim_{x \rightarrow 0} \frac{\sin \frac{x \times \pi}{180}}{x} \\
 &= \lim_{x \rightarrow 0} \frac{\sin \frac{\pi x}{180}}{x \times \frac{\pi}{180}} \times \frac{\pi}{180} \\
 &= \frac{\pi}{180} \lim_{x \rightarrow 0} \frac{\sin \frac{\pi x}{180}}{\frac{\pi x}{180}} \\
 &= \frac{\pi}{180} \lim_{x \rightarrow 0} \frac{\sin \pi x}{\pi x} \\
 &= \frac{\pi}{180} \times 1 = \frac{\pi}{180} \\
 &= \frac{\pi}{180}
 \end{aligned}$$

$$\left[\because 1^\circ = \frac{\pi}{180} \text{ radians} \right]$$

$$\left[\because \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \right]$$

Q3

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{x^2}{\sin x^2} \\ &= \lim_{x \rightarrow 0} \frac{1}{\frac{\sin x^2}{x^2}} \\ &= \frac{1}{\lim_{x \rightarrow 0} \frac{\sin x^2}{x^2}} \\ &= \frac{1}{1} \\ &= 1 \end{aligned}$$

$$\left[\because \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \right]$$

Q4

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{\sin x \cos x}{3x} \\ &= \frac{1}{3} \lim_{x \rightarrow 0} \frac{\sin x \cos x}{x} \\ &= \frac{1}{3} \lim_{x \rightarrow 0} \frac{\sin x}{x} \times \lim_{x \rightarrow 0} \cos x \\ &= \frac{1}{3} \times 1 \times 1 \\ &= \frac{1}{3} \end{aligned}$$

$$\left[\because \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1, \lim_{x \rightarrow 0} \cos x = \cos 0^\circ = 1 \right]$$

Q5

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{3 \sin x - 4 \sin^3 x}{x} \\ &= \lim_{x \rightarrow 0} \frac{\sin 3x}{x} \\ &= \lim_{x \rightarrow 0} \frac{\sin 3x}{3x} \times 3 \\ &= 3 \times \lim_{x \rightarrow 0} \frac{\sin 3x}{3x} \\ &= 3 \times \lim_{3x \rightarrow 0} \frac{\sin 3x}{3x} \\ &= 3 \times 1 \\ &= 3 \end{aligned}$$

$$\left[\because \sin 3x = 3 \sin x - 4 \sin^3 x \right]$$

$$\left[\because x \rightarrow 0, 3x \rightarrow 0 \right]$$

$$\left[\because \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \right]$$

Q6

$$\lim_{x \rightarrow 0} \frac{\tan 8x}{\sin 2x}$$

$$= \frac{\lim_{x \rightarrow 0} \tan 8x}{\lim_{x \rightarrow 0} \sin 2x}$$

$$= \frac{\lim_{x \rightarrow 0} \frac{\tan 8x}{8x} \times 8x}{\lim_{x \rightarrow 0} \frac{\sin 2x}{2x} \times 2x}$$

$$= \frac{\lim_{8x \rightarrow 0} \frac{\tan 8x}{8x} \times \frac{8x}{2x}}{\lim_{2x \rightarrow 0} \frac{\sin 2x}{2x}}$$

$$\left[\begin{array}{l} \because x \rightarrow 0 \\ 8x \rightarrow 0 \\ 2x \rightarrow 0 \end{array} \right]$$

$$= \frac{1 \times \frac{8}{2}}{1}$$

$$\left[\because \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \text{ and } \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1 \right]$$

$$= \frac{8}{2}$$

$$= 4$$

Q7

$$\lim_{x \rightarrow 0} \frac{\tan mx}{\tan nx}$$

$$= \frac{\lim_{x \rightarrow 0} \tan mx}{\lim_{x \rightarrow 0} \tan nx}$$

$$= \frac{\lim_{mx \rightarrow 0} \frac{\tan mx}{mx} \times mx}{\lim_{nx \rightarrow 0} \frac{\tan nx}{nx} \times nx}$$

$$[\because \text{If } x \rightarrow 0 \text{ then } mx \rightarrow 0 \text{ also } nx \rightarrow 0]$$

$$= \frac{1 \times m}{1 \times n}$$

$$\left[\because \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1 \right]$$

$$= \frac{m}{n}$$

Q8

$$\lim_{x \rightarrow 0} \frac{\sin 5x}{\tan 3x}$$

$$= \frac{\lim_{x \rightarrow 0} \sin 5x}{\lim_{3x \rightarrow 0} \tan 3x}$$

$$= \frac{\lim_{5x \rightarrow 0} \frac{\sin 5x}{5x} \times 5}{\lim_{3x \rightarrow 0} \frac{\tan 3x}{3x} \times 3}$$

$$= \frac{5}{3} \times 1$$

$$= \frac{5}{3}$$

$$[\because \text{If } x \rightarrow 0 \text{ then } 3x \rightarrow 0, 5x \rightarrow 0]$$

$$\left[\because \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \text{ also } \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1 \right]$$

Q9

$$\lim_{x \rightarrow 0} \frac{\sin x^\circ}{x^\circ}$$

$$= \lim_{x \rightarrow 0} \frac{\sin \frac{x \times \pi}{180}}{x \times \frac{\pi}{180}}$$

$$= \lim_{\frac{\pi x}{180} \rightarrow 0} \frac{\sin \frac{\pi x}{180}}{\frac{\pi x}{180}}$$

$$= 1$$

$$\left[\because 1^\circ = \frac{\pi}{180} \text{ radians} \right]$$

$$\left[\because \text{If } x \rightarrow 0 \text{ then } \frac{\pi x}{180} \rightarrow 0 \right]$$

$$\left[\because \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \right]$$

Q10

$$\lim_{x \rightarrow 0} \frac{7x \cos x - 3 \sin x}{4x + \tan x}$$

$$= \lim_{x \rightarrow 0} \frac{7 \cos x - \frac{3 \sin x}{x}}{4 + \frac{\tan x}{x}}$$

$$= \frac{\lim_{x \rightarrow 0} 7 \cos x - \lim_{x \rightarrow 0} \frac{3 \sin x}{x}}{\lim_{x \rightarrow 0} 4 + \lim_{x \rightarrow 0} \frac{\tan x}{x}}$$

$$= \frac{7 \times \lim_{x \rightarrow 0} \cos x - 3 \lim_{x \rightarrow 0} \frac{\sin x}{x}}{4 + \lim_{x \rightarrow 0} \frac{\tan x}{x}}$$

$$= \frac{7 \times 1 - 3 \times 1}{4 + 1}$$

$$= \frac{4}{5}$$

$$\left[\begin{array}{l} \because \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \\ \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1 \end{array} \right]$$

Q11

$$\lim_{x \rightarrow 0} \frac{\cos ax - \cos bx}{\cos cx - \cos dx} = \lim_{x \rightarrow 0} \frac{\left(-2 \sin\left(\frac{a+b}{2}\right)x \sin\left(\frac{a-b}{2}\right)x\right)}{-2 \sin\left(\frac{c+d}{2}\right)x \sin\left(\frac{c-d}{2}\right)x}$$

$$= \frac{\lim_{x \rightarrow 0} \sin\left(\frac{a+b}{2}\right)x \lim_{x \rightarrow 0} \sin\left(\frac{a-b}{2}\right)x}{\lim_{x \rightarrow 0} \sin\left(\frac{c+d}{2}\right)x \lim_{x \rightarrow 0} \sin\left(\frac{c-d}{2}\right)x}$$

$$= \frac{\left(\lim_{x \rightarrow 0} \frac{\sin\left(\frac{a+b}{2}\right)x}{\left(\frac{a+b}{2}\right)x} \times \left(\frac{a+b}{2}\right)x\right) \left(\lim_{x \rightarrow 0} \frac{\sin\left(\frac{a-b}{2}\right)x}{\left(\frac{a-b}{2}\right)x} \times \left(\frac{a-b}{2}\right)x\right)}{\left(\lim_{x \rightarrow 0} \frac{\sin\left(\frac{c+d}{2}\right)x}{\left(\frac{c+d}{2}\right)x} \times \left(\frac{c+d}{2}\right)x\right) \left(\lim_{x \rightarrow 0} \frac{\sin\left(\frac{c-d}{2}\right)x}{\left(\frac{c-d}{2}\right)x} \times \left(\frac{c-d}{2}\right)x\right)}$$

$$= \frac{(a+b)(a-b)}{(c+d)(c-d)}$$

$$\left[\because \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \right]$$

$$= \frac{a^2 - b^2}{c^2 - d^2}$$

Q12

$$\lim_{x \rightarrow 0} \frac{\tan^2 3x}{x^2}$$

$$= \left(\lim_{x \rightarrow 0} \frac{\tan 3x}{x} \right)^2$$

$$= \left(\lim_{x \rightarrow 0} \frac{\tan 3x}{3x} \right)^2 \times 9$$

$$= 1 \times 9$$

$$\left[\because \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1 \right]$$

$$= 9$$

Q13

$$\begin{aligned}
 & \lim_{x \rightarrow 0} \frac{1 - \cos mx}{x^2} \\
 &= \lim_{x \rightarrow 0} \frac{2 \sin^2 \frac{mx}{2}}{x^2} \\
 &= 2 \times \left(\lim_{x \rightarrow 0} \frac{\sin \frac{mx}{2}}{x} \right)^2 \\
 &= 2 \times \left(\lim_{x \rightarrow 0} \frac{\sin \frac{mx}{2}}{\frac{mx}{2}} \right)^2 \times \left(\frac{m}{2} \right)^2 \\
 &= 2 \times \frac{m^2}{4} = \frac{m^2}{2} \\
 &= \frac{m^2}{2}
 \end{aligned}$$

$$\left[\because \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \right]$$

Q14

$$\begin{aligned}
 & \lim_{x \rightarrow 0} \frac{3 \sin 2x + 2x}{3x + 2 \tan 3x} \\
 &= \lim_{x \rightarrow 0} \frac{3 \frac{\sin 2x}{x} + 2}{3 + 2 \tan \frac{3x}{x}} \\
 &= \frac{\lim_{x \rightarrow 0} \frac{3 \sin 2x}{x} + \lim_{x \rightarrow 0} 2}{\lim_{x \rightarrow 0} 3 + \lim_{x \rightarrow 0} \frac{2 \tan 3x}{x}} \\
 &= \frac{\left(3 \lim_{2x \rightarrow 0} \frac{\sin 2x}{2x} \times 2 \right) + 2}{3 + \left(2 \times \lim_{3x \rightarrow 0} \frac{\tan 3x}{3x} \times 3 \right)} \\
 &= \frac{(3 \times 2) + 2}{3 + (2 \times 3)} \\
 &= \frac{8}{9}
 \end{aligned}$$

$$\left[\because \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \text{ also } \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1 \right]$$

Q15

$$\begin{aligned}
& \lim_{x \rightarrow 0} \frac{\cos 3x - \cos 7x}{x^2} \\
&= \lim_{x \rightarrow 0} \left(\frac{-2 \sin \left(\frac{3x + 7x}{2} \right) \sin \left(\frac{3x - 7x}{2} \right)}{x^2} \right) \\
&= \lim_{x \rightarrow 0} \left(\frac{-2 \sin 5x \sin \left(\frac{-4x}{2} \right)}{x^2} \right) \\
&= \left(\lim_{x \rightarrow 0} \frac{-2 \sin 5x}{x} \right) \times \left(\lim_{x \rightarrow 0} \frac{\sin(-2x)}{x} \right) \\
&= \left(-2 \lim_{x \rightarrow 0} \frac{\sin 5x}{5x} \times 5 \right) \times \left(-1 \lim_{x \rightarrow 0} \frac{\sin 2x}{2x} \times 2 \right) \\
&= (-2 \times 5)(-1 \times 2) \\
&= 20
\end{aligned}$$

Q16

$$\begin{aligned}
& \frac{\lim_{\theta \rightarrow 0} \sin 3\theta}{\lim_{\theta \rightarrow 0} \tan 2\theta} \\
&= \frac{\lim_{3\theta \rightarrow 0} \frac{\sin 3\theta}{3\theta} \times 3\theta}{\lim_{2\theta \rightarrow 0} \frac{\tan 2\theta}{2\theta} \times 2\theta} \\
&= \frac{\left(\lim_{3\theta \rightarrow 0} \frac{\sin 3\theta}{3\theta} \right) \times 3\theta}{\left(\lim_{2\theta \rightarrow 0} \frac{\sin 2\theta}{2\theta} \right) \times 2\theta} \\
&= \frac{1}{1} \times \frac{3}{2} \qquad \left[\because \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \text{ and } \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1 \right] \\
&= \frac{3}{2}
\end{aligned}$$

Q17

$$\begin{aligned}
& \lim_{x \rightarrow 0} \frac{\sin x^2 (1 - \cos x^2)}{x^6} \\
&= \lim_{x \rightarrow 0} \frac{\sin x^2 \times 2 \sin^2 \frac{x^2}{2}}{x^6} \\
&= \lim_{x \rightarrow 0} \frac{\sin x^2}{x^2} \times \lim_{x \rightarrow 0} \frac{2 \sin^2 \frac{x^2}{2}}{x^4} \\
&= \left(\lim_{x \rightarrow 0} \frac{\sin x}{x} \right)^2 \times 2 \times \left(\lim_{x \rightarrow 0} \frac{\sin \frac{x^2}{2}}{\frac{x^2}{2}} \right)^2 \times \frac{1}{4} \\
&= (1)^2 \times 2 \times 1 \times \frac{1}{4} \\
&= \frac{1}{2}
\end{aligned}$$

$$\left[\because \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \right]$$

Q18

$$\begin{aligned}
& \lim_{x \rightarrow 0} \frac{\sin^2 4x^2}{x^4} \\
&= \lim_{x \rightarrow 0} \frac{(\sin 4x^2)^2}{x^4} \\
&= \lim_{x \rightarrow 0} \frac{(\sin 4x^2)^2}{(x^2)^2} \\
&= \left(\lim_{x \rightarrow 0} \frac{\sin 4x^2}{x^2} \right)^2 \\
&= \left(\lim_{4x^2 \rightarrow 0} \frac{\sin 4x^2}{4x^2} \right) \times 16 \\
&= 1 \times 16 = 16 \\
&= 16
\end{aligned}$$

$$\left[\because \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \right]$$

Q19

$$\lim_{x \rightarrow 0} \frac{x \cos x + 2 \sin x}{x^2 + \tan x}$$

Dividing each term by x

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{\cos x + \frac{2 \sin x}{x}}{x + \frac{\tan x}{x}} \\ &= \frac{\lim_{x \rightarrow 0} \cos x + \lim_{x \rightarrow 0} \frac{2 \sin x}{x}}{\lim_{x \rightarrow 0} x + \lim_{x \rightarrow 0} \frac{\tan x}{x}} \\ &= \frac{\cos 0 + 2 \lim_{x \rightarrow 0} \frac{\sin x}{x}}{0 + \lim_{x \rightarrow 0} \frac{\tan x}{x}} \\ &= \frac{1+2}{0+1} = 3 \\ &= 3 \end{aligned}$$

$$\left[\because \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1, \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1 \right]$$

Q20

$$\lim_{x \rightarrow 0} \frac{2x - \sin x}{\tan x + x}$$

Dividing each term by x

$$\begin{aligned} &= \lim_{x \rightarrow 0} \frac{2 - \frac{\sin x}{x}}{\frac{\tan x}{x} + 1} \\ &= \frac{\lim_{x \rightarrow 0} 2 - \lim_{x \rightarrow 0} \frac{\sin x}{x}}{\lim_{x \rightarrow 0} \frac{\tan x}{x} + \lim_{x \rightarrow 0} 1} \\ &= \frac{2-1}{1+1} \\ &= \frac{1}{2} \end{aligned}$$

$$\left[\because \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1, \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1 \right]$$

Q21

$$\lim_{x \rightarrow 0} \frac{5x \cos x + 3 \sin x}{3x^2 + \tan x}$$

$$\lim_{x \rightarrow 0} \frac{5 \cos x + 3 \frac{\sin x}{x}}{3x + \frac{\tan x}{x}}$$

$$= \frac{\lim_{x \rightarrow 0} 5 \cos x + \lim_{x \rightarrow 0} \frac{3 \sin x}{x}}{\lim_{x \rightarrow 0} 3x + \lim_{x \rightarrow 0} \frac{\tan x}{x}}$$

$$= \frac{5 \lim_{x \rightarrow 0} \cos x + 3 \lim_{x \rightarrow 0} \frac{\sin x}{x}}{3 \lim_{x \rightarrow 0} x + \lim_{x \rightarrow 0} \frac{\tan x}{x}}$$

$$= \frac{5 \times \cos 0 + 3 \times 1}{3 \times 0 + 1}$$

$$= \frac{5+3}{1}$$

$$= 8$$

$$\left[\because \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \text{ and } \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1 \right]$$

Q22

$$\lim_{x \rightarrow 0} \frac{\sin 3x - \sin x}{\sin x}$$

$$= \lim_{x \rightarrow 0} \left(\frac{2 \cos \left(\frac{3x+x}{2} \right) \sin \left(\frac{3x-x}{2} \right)}{\sin x} \right)$$

$$= \lim_{x \rightarrow 0} \left(\frac{2 \cos 2x \sin x}{\sin x} \right)$$

$$= \lim_{x \rightarrow 0} (2 \cos x)$$

$$= 2 \lim_{x \rightarrow 0} \cos 2x$$

$$= 2 \times \cos 0$$

$$= 2 \times 1 = 2$$

$$= 2$$

Q23

$$\begin{aligned}
& \lim_{x \rightarrow 0} \frac{\sin 5x - \sin 3x}{\sin x} \\
&= \lim_{x \rightarrow 0} \frac{2 \cos \left(\frac{5x + 3x}{2} \right) \sin \left(\frac{5x - 3x}{2} \right)}{\sin x} \\
&= \lim_{x \rightarrow 0} \frac{2 \cos 4x \sin x}{\sin x} \\
&= 2 \lim_{x \rightarrow 0} \cos 4x \\
&= 2 \times \cos 0 \\
&= 2
\end{aligned}$$

Q24

$$\begin{aligned}
& \lim_{x \rightarrow 0} \frac{\cos 3x - \cos 5x}{x^2} \\
&= \lim_{x \rightarrow 0} \frac{\left(-2 \sin \left(\frac{3x + 5x}{2} \right) \sin \left(\frac{3x - 5x}{2} \right) \right)}{x^2} \\
&= \lim_{x \rightarrow 0} \left(\frac{-2 \sin 4x \sin(-x)}{x^2} \right) \\
&= \lim_{x \rightarrow 0} \frac{2 \sin 4x \sin x}{x^2} \\
&= \lim_{x \rightarrow 0} \frac{2 \sin 4x}{x} \times \lim_{x \rightarrow 0} \frac{\sin x}{x} \\
&= \left(2 \lim_{x \rightarrow 0} \frac{\sin 4x}{4x} \times 4 \right) \times \left(\lim_{x \rightarrow 0} \frac{\sin x}{x} \right) \quad \left[\because \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \right] \\
&= 8
\end{aligned}$$

Q25

$$\lim_{x \rightarrow 0} \frac{\frac{\tan 3x}{x} - \frac{\tan 2x}{x}}{\frac{3x}{x} - \frac{\sin^2 x}{x}}$$

$$= \frac{\left(\lim_{x \rightarrow 0} \frac{\tan 3x}{3x} \times 3 \right) - \left(\lim_{x \rightarrow 0} \frac{\tan 2x}{2x} \times 2 \right)}{\left(\lim_{x \rightarrow 0} \frac{3x}{2} \right) - \lim_{x \rightarrow 0} \frac{(\sin x)^2}{x}}$$

$$= \frac{3 - 2}{3 - \left(\frac{\sin x}{x} \right)^2 \times x}$$

$$= \frac{3 - 2}{3 - 0} = \frac{1}{3}$$

$$= \frac{1}{3}$$

$$\left[\because \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1 \right]$$

$$\left[\because \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \right]$$

Q26

$$\lim_{x \rightarrow 0} \frac{\sin(2+x) - \sin(2-x)}{x}$$

$$= \lim_{x \rightarrow 0} 2 \cos \left(\frac{2+x+2-x}{2} \right) \times \sin \left(\frac{2+x-2+x}{2} \right)$$

$$= 2 \lim_{x \rightarrow 0} \frac{\cos(2) \times \sin x}{x}$$

$$= 2 \lim_{x \rightarrow 0} \cos 2 \times \lim_{x \rightarrow 0} \frac{\sin x}{x}$$

$$= 2 \cos 2 \times 1$$

$$= 2 \cos 2$$

$$\left[\because \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \right]$$

Q27

$$\begin{aligned}& \lim_{h \rightarrow 0} \frac{(a+h)^2 \sin(a+h) - a^2 \sin a}{h} \\&= \lim_{h \rightarrow 0} \frac{(a^2 + h^2 + 2ah) \sin(a+h) - a^2 \sin a}{h} \\&= \lim_{h \rightarrow 0} \frac{a^2 \sin(a+h) + h^2 \sin(a+h) + 2ah \sin(a+h) - a^2 \sin a}{h} \\&= \lim_{h \rightarrow 0} \left[\frac{a^2 (\sin(a+h) - \sin a)}{h} + \frac{h^2 \sin(a+h)}{h(a+h)} \times (a+h) + \frac{2ah}{h} (\sin(a+h)) \right] \\&= \left[a^2 \lim_{h \rightarrow 0} \frac{2 \cos\left(\frac{a+h+a}{2}\right) \sin\left(\frac{a+h-a}{2}\right)}{h} \right] + [0] + 2a \lim_{h \rightarrow 0} \sin(a+h) \\&= \left(2a^2 \lim_{h \rightarrow 0} \cos\left(a + \frac{h}{2}\right) \frac{\sin \frac{h}{2}}{\frac{h}{2}} \times \frac{1}{2} \right) + (2a \times \sin a) \\&= \left(2a^2 \cos a \times \frac{1}{2} \right) + (2a \sin a) \\&= a^2 \cos a + 2a \sin a\end{aligned}$$

Q28

$$\lim_{x \rightarrow 0} \frac{\tan x - \sin x}{\sin 3x - 3 \sin x}$$

$$\lim_{x \rightarrow 0} \frac{\frac{\sin x}{\cos x} - \sin x}{3 \sin x - 4 \sin^3 x - 3 \sin x}$$

$$= \lim_{x \rightarrow 0} \frac{\sin x \left(\frac{1}{\cos x} - 1 \right)}{-4 \sin^3 x}$$

$$= \frac{-1}{4} \lim_{x \rightarrow 0} \frac{\frac{1}{\cos x} - 1}{\sin^2 x}$$

$$= \frac{-1}{4} \lim_{x \rightarrow 0} \frac{\frac{1 - \cos x}{\cos x}}{1 - \cos^2 x}$$

$$= \frac{-1}{4} \lim_{x \rightarrow 0} \frac{1 - \cos x}{(\cos x)(1 - \cos x)(1 + \cos x)}$$

$$= \frac{-1}{4} \lim_{x \rightarrow 0} \frac{1}{(\cos x)(1 + \cos x)}$$

$$= \frac{-1}{4} \times \frac{1}{1(1+1)}$$

$$= \frac{-1}{4} \times \frac{1}{2}$$

$$= -\frac{1}{8}$$

Q29

$$\lim_{x \rightarrow 0} \frac{\sec 5x - \sec 3x}{\sec 3x - \sec x} = \lim_{x \rightarrow 0} \left(\frac{\cos 3x - \cos 5x}{\cos 3x \cos 5x} \times \frac{\cos x \cos 3x}{\cos x - \cos 3x} \right)$$

$$= \lim_{x \rightarrow 0} \left(\frac{\cos 3x - \cos 5x}{\cos x - \cos 3x} \times \frac{\cos x \cos 3x}{\cos 3x \cos 5x} \right)$$

$$= \lim_{x \rightarrow 0} \left(\frac{-2 \sin 4x \sin(-x)}{-2 \sin(2x) \sin(-x)} \times \frac{\cos x}{\cos 5x} \right)$$

$$= \lim_{x \rightarrow 0} \left(\frac{\sin 4x}{\sin 2x} \times \frac{\cos x}{\cos 5x} \right)$$

$$= \frac{\lim_{x \rightarrow 0} \sin 4x \times \lim_{x \rightarrow 0} \cos x}{\lim_{x \rightarrow 0} \sin 2x \times \lim_{x \rightarrow 0} \cos 5x}$$

$$= \frac{\left(\lim_{4x \rightarrow 0} \frac{\sin 4x}{4x} \times 4x \right) \times \left(\lim_{x \rightarrow 0} \cos x \right)}{\left(\lim_{2x \rightarrow 0} \frac{\sin 2x}{2x} \times 2x \right) \times \left(\lim_{x \rightarrow 0} \cos 5x \right)}$$

$$= \frac{(1 \times 4x) \times 1}{1 \times 2x \times 1}$$

$$= \frac{4x}{2x}$$

$$= 2$$

$$\left[\because \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \text{ and } \lim_{x \rightarrow 0} \cos x = \cos 0 = 1 \right]$$

Q30

$$\begin{aligned}
 & \lim_{x \rightarrow 0} \frac{1 - \cos 2x}{\cos 2x - \cos 8x} \\
 &= \lim_{x \rightarrow 0} \frac{2 \sin^2 x}{-2 \sin\left(\frac{2x+8x}{2}\right) \sin\left(\frac{2x-8x}{2}\right)} \\
 &= \lim_{x \rightarrow 0} \frac{\sin^2 x}{\sin 5x \times \sin(-3x)} \\
 &= \frac{\lim_{x \rightarrow 0} \sin^2 x}{-\left(\lim_{x \rightarrow 0} \sin 5x\right) \left(-\lim_{x \rightarrow 0} \sin 3x\right)} \\
 &= \frac{\left(\lim_{x \rightarrow 0} \frac{\sin x}{x}\right)^2 \times x^2}{\left(\lim_{x \rightarrow 0} \frac{\sin 5x}{5x}\right) \times 5x \left(\lim_{x \rightarrow 0} \frac{\sin 3x}{3x}\right) \times 3x} \\
 &= \frac{1 \times x^2}{1 \times 5x \times 1 \times 3x} \\
 &= \frac{x^2}{15x^2} \\
 &= \frac{1}{15}
 \end{aligned}$$

$$\left[\because \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \right]$$

Q31

$$\begin{aligned}
 & \lim_{x \rightarrow 0} \frac{1 - \cos 2x + \tan^2 x}{x \sin x} \\
 &= \lim_{x \rightarrow 0} \frac{2 \sin^2 x + \tan^2 x}{x \sin x} \\
 &= \frac{2 \lim_{x \rightarrow 0} \sin^2 x + \lim_{x \rightarrow 0} \tan^2 x}{\lim_{x \rightarrow 0} x \sin x} \\
 &= \frac{\left(2 \left(\lim_{x \rightarrow 0} \frac{\sin x}{x}\right)^2 \times x^2\right) + \left(\lim_{x \rightarrow 0} \frac{\tan x}{x}\right)^2 \times x^2}{\left(\lim_{x \rightarrow 0} \frac{\sin x}{x}\right) \times x^2} \\
 &= \frac{(2 \times 1 \times x^2) + (1 \times x^2)}{(1 \times x^2)} \\
 &= \frac{3x^2}{x^2} \\
 &= 3
 \end{aligned}$$

$$\left[\because \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \text{ and } \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1 \right]$$

Q32

$$\begin{aligned}
 & \lim_{x \rightarrow 0} \frac{\sin(a+x) + \sin(a-x) - 2\sin a}{x \sin x} \\
 &= \lim_{x \rightarrow 0} \frac{2 \sin\left(\frac{a+x+a-x}{2}\right) \cos\left(\frac{a+x-a-x}{2}\right) - 2\sin a}{x \sin x} \\
 &= \lim_{x \rightarrow 0} \frac{2 \sin a (\cos x - 1)}{x \sin x} \\
 &= \lim_{x \rightarrow 0} \frac{-2 \sin a (1 - \cos x)}{x \sin x} \\
 &= -2 \sin a \lim_{x \rightarrow 0} \frac{2 \sin^2 \frac{x}{2}}{\left(2 \sin \frac{x}{2} \cos \frac{x}{2}\right) x} \\
 &= -2 \sin a \lim_{x \rightarrow 0} \frac{2 \sin \frac{x}{2}}{\left(\cos \frac{x}{2}\right) x} \\
 &= -2 \sin a \lim_{x \rightarrow 0} \frac{\tan \frac{x}{2}}{\frac{x}{2}} \times \frac{1}{2} \\
 &= -2 \sin a \times 1 \times \frac{1}{2} \\
 &= -\sin a \\
 &= -\sin a
 \end{aligned}$$

$$\left[\because \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1 \right]$$

Q33

$$\begin{aligned}
 & \lim_{x \rightarrow 0} \frac{x^2 - \tan 2x}{\tan x} \\
 &= \lim_{x \rightarrow 0} \frac{\left(\frac{x^2}{2x} - \frac{\tan 2x}{2x}\right) \times 2x}{\frac{\tan x}{x} \times x} \\
 &= \lim_{x \rightarrow 0} \frac{2\left(\frac{x}{2} - \frac{\tan 2x}{2x}\right)}{\frac{\tan x}{x}} \\
 &= 2\left(\frac{0-1}{1}\right) \\
 &= -2
 \end{aligned}$$

Q34

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{\sqrt{2} + \sqrt{1 + \cos x}}{\sin^2 x} &= \lim_{x \rightarrow 0} \frac{\sqrt{2} + \sqrt{1 + \cos x}}{\sin^2 x} \times \frac{\sqrt{2} - \sqrt{1 + \cos x}}{\sqrt{2} - \sqrt{1 + \cos x}} \\
 &= \lim_{x \rightarrow 0} \frac{2 - 1 - \cos x}{\sin^2 x} \\
 &= \lim_{x \rightarrow 0} \frac{1 - \cos x}{\sin^2 x} \\
 &= \lim_{x \rightarrow 0} \frac{1 - \cos x}{1 - \cos^2 x} \\
 &= \lim_{x \rightarrow 0} \frac{1}{1 + \cos x} \\
 &= \frac{1}{1 + 1} \\
 &= \frac{1}{2}
 \end{aligned}$$

Q35

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{x \tan x}{1 - \cos x} &= \lim_{x \rightarrow 0} \frac{x \sin x}{\cos x (1 - \cos x)} \\
 &= \lim_{x \rightarrow 0} \frac{x \sin x}{\cos x \left(2 \sin^2 \frac{x}{2} \right)} \\
 &= \lim_{x \rightarrow 0} \frac{x \left(2 \sin \frac{x}{2} \cos \frac{x}{2} \right)}{\cos x \left(2 \sin^2 \frac{x}{2} \right)} \\
 &= \lim_{x \rightarrow 0} \frac{x \cos \frac{x}{2}}{\cos x \frac{\sin x}{2}} \\
 &= \lim_{x \rightarrow 0} \frac{1}{\frac{\tan x}{\cos x \times \frac{2}{x}}} \\
 &= \lim_{x \rightarrow 0} \frac{1}{\cos x} \times \frac{1}{\frac{\tan x}{\frac{2}{x}} \times \frac{1}{2}} \\
 &= 1 \times 1 \times 2 \\
 &= 2
 \end{aligned}$$

Q36

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{x^2 + 1 - \cos x}{x \sin x} \\ &= \lim_{x \rightarrow 0} \frac{x^2 + 2 \sin^2 \frac{x}{2}}{x \sin x} \\ &= \lim_{x \rightarrow 0} \frac{x^2 \left[1 + 2 \left(\frac{\sin \frac{x}{2}}{x} \right)^2 \right]}{x \sin x} \\ &= \lim_{x \rightarrow 0} \frac{1 + 2 \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}} \right)^2 \times \frac{1}{4}}{\frac{\sin x}{x}} \\ &= \lim_{x \rightarrow 0} \frac{1 + \lim_{x \rightarrow 0} 2 \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}} \right)^2 \times \frac{1}{4}}{\lim_{x \rightarrow 0} \frac{\sin x}{x}} \\ &= \frac{1 + 2 \times 1 \times \frac{1}{4}}{1} = \frac{1 + \frac{1}{2}}{1} \\ &= \frac{3}{2} \end{aligned}$$

Q37

$$\begin{aligned}
 & \lim_{x \rightarrow 0} \frac{\sin 2x (\cos 3x - \cos x)}{x^3} \\
 &= \lim_{x \rightarrow 0} \frac{\sin 2x \left(-2 \sin \left(\frac{3x+x}{2} \right) \sin \left(\frac{3x-x}{2} \right) \right)}{x^3} \\
 &= \lim_{x \rightarrow 0} \frac{\sin 2x (-2 \sin 2x \sin x)}{x^3} \\
 &= \frac{-2 \lim_{x \rightarrow 0} \sin 2x \times \lim_{x \rightarrow 0} \sin 2x \times \lim_{x \rightarrow 0} \sin x}{x^3} \\
 &= -2 \left(\lim_{x \rightarrow 0} \frac{\sin 2x}{2x} \times 2 \right) \times \left(2 \lim_{x \rightarrow 0} \frac{\sin 2x}{2x} \right) \times \left(\lim_{x \rightarrow 0} \frac{\sin x}{x} \right) \\
 &= -2(1 \times 2) \times (2) \times (1) \\
 &= -8
 \end{aligned}$$

Q38

$$\begin{aligned}
 & \lim_{x \rightarrow 0} \frac{2 \sin x^\circ - \sin 2x^\circ}{x^3} \\
 &= \lim_{x \rightarrow 0} \frac{2 \sin \frac{\pi x}{180} - \sin \frac{2\pi x}{180}}{x^3} \\
 &= \lim_{x \rightarrow 0} \frac{2 \sin \frac{\pi x}{180} - 2 \sin \frac{\pi x}{180} \cos \frac{\pi x}{180}}{x^3} \\
 &= \lim_{x \rightarrow 0} \frac{2 \sin \frac{\pi x}{180} \left(2 \sin^2 \frac{\pi x}{360} \right)}{x^3} \\
 &= 4 \left(\lim_{x \rightarrow 0} \frac{\sin \frac{\pi x}{180}}{x} \right) \times \left(\lim_{x \rightarrow 0} \frac{\sin \frac{\pi x}{360}}{x} \right) \times \left(\lim_{x \rightarrow 0} \frac{\sin \frac{\pi x}{360}}{x} \right) \\
 &= 4 \left(\lim_{x \rightarrow 0} \frac{\sin \frac{\pi x}{180}}{\frac{\pi x}{180}} \times \frac{\pi}{180} \right) \times \left(\lim_{x \rightarrow 0} \frac{\sin \frac{\pi x}{360}}{\frac{\pi x}{360}} \times \frac{\pi}{360} \right) \times \left(\lim_{x \rightarrow 0} \frac{\sin \frac{\pi x}{360}}{\frac{\pi x}{360}} \times \frac{\pi}{360} \right) \\
 &= 4 \times \frac{\pi}{180} \times \frac{\pi}{360} \times \frac{\pi}{360} \\
 &= \left(\frac{\pi}{180} \right)^3
 \end{aligned}$$

Q39

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{x^3 \cot x}{1 - \cos x} \\ &= \lim_{x \rightarrow 0} \frac{x^3}{\tan x (1 - \cos x)} \\ &= \lim_{x \rightarrow 0} \frac{x^3}{\tan x \cdot 2 \sin^2 \frac{x}{2}} \\ &= \lim_{x \rightarrow 0} \frac{1}{\frac{\tan x}{x} \times \frac{2 \sin^2 \frac{x}{2}}{x^2}} \end{aligned}$$

$$= \frac{1}{\left(\lim_{x \rightarrow 0} \frac{\tan x}{x} \right) \times 2 \left(\lim_{x \rightarrow 0} \frac{\sin \frac{x}{2}}{\frac{x}{2}} \right)^2 \times \frac{1}{4}}$$

$$= \frac{1}{1 \times 2 \times 1 \times \frac{1}{4}}$$

$$= 2$$

$$\left[\because \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1, \lim_{\theta \rightarrow 0} \frac{\tan \theta}{\theta} = 1 \right]$$

Q40

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{x \tan x}{1 - \cos 2x} \\ &= \lim_{x \rightarrow 0} \frac{x \tan x}{2 \sin^2 x} \end{aligned}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{\tan x}{x}}{\frac{2 \sin^2 x}{x^2}}$$

$$= \frac{\lim_{x \rightarrow 0} \frac{\tan x}{x}}{2 \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right)^2}$$

$$= \frac{1}{2 \times 1}$$

$$= \frac{1}{2}$$

$$\left[\because \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1, \lim_{\theta \rightarrow 0} \frac{\tan \theta}{\theta} = 1 \right]$$

Q41

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{\sin(3+x) - \sin(3-x)}{x} \\ &= \lim_{x \rightarrow 0} \frac{2 \cos\left(\frac{3+x+3-x}{2}\right) \sin\left(\frac{3+x-3+x}{2}\right)}{x} \\ &= 2 \lim_{x \rightarrow 0} \frac{\cos 3 \cdot \sin x}{x} \\ &= 2 \cos 3 \lim_{x \rightarrow 0} \frac{\sin x}{x} \\ &= 2 \cos 3 \times 1 \\ &= 2 \cos 3 \end{aligned}$$

Q42

$$\lim_{x \rightarrow 0} kx \operatorname{cosec} x = \lim_{x \rightarrow 0} x \operatorname{cosec} kx$$

$$\text{LHS} = \lim_{x \rightarrow 0} kx \operatorname{cosec} x = k \lim_{x \rightarrow 0} x \operatorname{cosec} x$$

$$= k \lim_{x \rightarrow 0} \frac{x}{\sin x}$$

$$= k \lim_{x \rightarrow 0} \frac{1}{\frac{\sin x}{x}}$$

$$= k \times 1$$

$$\left[\because \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \right]$$

$$\text{Also, RHS} = \lim_{x \rightarrow 0} x \operatorname{cosec} kx$$

$$= \lim_{x \rightarrow 0} \frac{x}{\sin kx}$$

$$= \lim_{x \rightarrow 0} \frac{x}{\frac{\sin kx}{kx} \times kx}$$

$$= \frac{1}{k}$$

$$\left[\because \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \right]$$

$$\text{As, LHS} = \text{RHS}$$

$$\Rightarrow k = \frac{1}{k}$$

$$k^2 = 1$$

$$k = \pm 1$$

Q43

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{3 \sin^2 x - 2 \sin x^2}{3x^2} \\ &= \lim_{x \rightarrow 0} \frac{3 \sin^2 x}{3x^2} - \lim_{x \rightarrow 0} \frac{2 \sin x^2}{3x^2} \\ &= \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right)^2 - \frac{2}{3} \lim_{x \rightarrow 0} \frac{\sin x^2}{x^2} \\ &= 1 - \frac{2}{3} \times 1 \qquad \left[\because \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \right] \\ &= 1 - \frac{2}{3} \\ &= \frac{3-2}{3} \\ &= \frac{1}{3} \end{aligned}$$

Q44

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{\sqrt{1+\sin x} - \sqrt{1-\sin x}}{x} \\ &= \lim_{x \rightarrow 0} \frac{(\sqrt{1+\sin x} - \sqrt{1-\sin x})(\sqrt{1+\sin x} + \sqrt{1-\sin x})}{x(\sqrt{1+\sin x} + \sqrt{1-\sin x})} \\ &= \lim_{x \rightarrow 0} \frac{((1+\sin x) - (1-\sin x))}{x(\sqrt{1+\sin x} + \sqrt{1-\sin x})} \\ &= \lim_{x \rightarrow 0} \frac{2 \sin x}{x(\sqrt{1+\sin x} + \sqrt{1-\sin x})} \\ &= 2 \lim_{x \rightarrow 0} \frac{\sin x}{x} \times \frac{1}{\lim_{x \rightarrow 0} (\sqrt{1+\sin x} + \sqrt{1-\sin x})} \\ &= 2 \times 1 \times \frac{1}{2} \\ &= 1 \end{aligned}$$

Q45

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{1 - \cos 4x}{x^2} \\ &= \lim_{x \rightarrow 0} \frac{2 \sin^2 2x}{x^2} \\ &= 2 \lim_{x \rightarrow 0} \left(\frac{\sin 2x}{x} \right)^2 \\ &= 2 \lim_{x \rightarrow 0} \left(\frac{\sin 2x}{2x} \right)^2 \times (2)^2 \\ &= 2 \times 1 \times 4 \\ &= 8 \end{aligned}$$

Q46

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{x \cos x + \sin x}{x^2 + \tan x} \\ &= \lim_{x \rightarrow 0} \frac{\cos x + \frac{\sin x}{x}}{x + \frac{\tan x}{x}} \\ &= \frac{\lim_{x \rightarrow 0} \cos x + \lim_{x \rightarrow 0} \frac{\sin x}{x}}{\lim_{x \rightarrow 0} x + \lim_{x \rightarrow 0} \frac{\tan x}{x}} \\ &= \frac{1+1}{0+1} = \frac{2}{1} \\ &= 2 \end{aligned}$$

Q47

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{1 - \cos 2x}{3 \tan^2 x} \\ &= \lim_{x \rightarrow 0} \frac{2 \sin^2 x}{3 \tan^2 x} \\ &= \frac{2}{3} \lim_{x \rightarrow 0} \frac{\sin^2 x}{\left(\frac{\sin^2 x}{\cos^2 x} \right)} \\ &= \frac{2}{3} \lim_{x \rightarrow 0} \cos^2 x \\ &= \frac{2}{3} \end{aligned}$$

Q48

$$\begin{aligned} & \lim_{\theta \rightarrow 0} \frac{1 - \cos 4\theta}{1 - \cos 6\theta} \\ &= \lim_{\theta \rightarrow 0} \frac{2 \sin^2 2\theta}{2 \sin^2 3\theta} \\ &= \lim_{\theta \rightarrow 0} \frac{(\sin 2\theta)^2}{(\sin 3\theta)^2} \\ &= \frac{\lim_{\theta \rightarrow 0} \left(\frac{\sin 2\theta}{2\theta} \right)^2 \times 4\theta^2}{\lim_{\theta \rightarrow 0} \left(\frac{\sin 3\theta}{3\theta} \right)^2 \times 9\theta^2} \\ &= \frac{1 \times 4\theta^2}{1 \times 9\theta^2} \\ &= \frac{4}{9} \end{aligned}$$

Q49

$$\lim_{x \rightarrow 0} \frac{ax + x \cos x}{b \sin x}$$

$$= \lim_{x \rightarrow 0} \frac{a + \cos x}{\frac{b \sin x}{x}}$$

$$= \frac{\lim_{x \rightarrow 0} a + \lim_{x \rightarrow 0} \cos x}{\lim_{x \rightarrow 0} \frac{b \sin x}{x}}$$

$$= \frac{a+1}{b}$$

$$\left[\because \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \right]$$

$$= \frac{a+1}{b}$$

Q50

$$\lim_{\theta \rightarrow 0} \frac{\sin 4\theta}{\tan 3\theta}$$

$$= \frac{\lim_{\theta \rightarrow 0} \sin 4\theta}{\lim_{\theta \rightarrow 0} \tan 3\theta}$$

$$= \frac{\left(\lim_{\theta \rightarrow 0} \frac{\sin 4\theta}{4\theta} \right) \times 4\theta}{\left(\lim_{\theta \rightarrow 0} \frac{\tan 3\theta}{3\theta} \right) \times 3\theta}$$

$$= \frac{1 \times 4\theta}{1 \times 3\theta}$$

$$= \frac{4}{3}$$

Q51

$$\begin{aligned}
\lim_{x \rightarrow 0} \frac{2 \sin x - \sin 2x}{x^3} &= \lim_{x \rightarrow 0} \frac{2 \sin x - 2 \sin x \cos x}{x^3} \\
&= \lim_{x \rightarrow 0} \frac{2 \sin x (1 - \cos x)}{x^3} \\
&= \lim_{x \rightarrow 0} \frac{2 \sin x (1 - \cos x)}{x^3} \times \frac{1 + \cos x}{1 + \cos x} \\
&= \lim_{x \rightarrow 0} \frac{2 \sin x (1 - \cos^2 x)}{x^3 (1 + \cos x)} \\
&= \lim_{x \rightarrow 0} \frac{2 \sin x (\sin^2 x)}{x^3 (1 + \cos x)} \\
&= \lim_{x \rightarrow 0} \frac{2 \sin^3 x}{x^3 (1 + \cos x)} \\
&= 2 \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right)^3 \times \lim_{x \rightarrow 0} \frac{1}{1 + \cos x} \\
&= 2 \times 1 \times \frac{1}{(1 + 1)} \\
&= 1
\end{aligned}$$

Q52

$$\begin{aligned}
&\lim_{x \rightarrow 0} \frac{1 - \cos 5x}{1 - \cos 6x} \\
&= \frac{\lim_{x \rightarrow 0} 2 \sin^2 \frac{5x}{2}}{\lim_{x \rightarrow 0} 2 \sin^2 3x} \\
&= \frac{2 \left(\lim_{x \rightarrow 0} \frac{\sin \frac{5x}{2}}{\frac{5x}{2}} \right)^2 \times \frac{25}{4} x^2}{2 \left(\lim_{x \rightarrow 0} \frac{\sin 3x}{3x} \right)^2 \times 9x^2} \\
&= \frac{2 \times 1 \times \frac{25}{4} x^2}{2 \times 1 \times 9x^2} \\
&= \frac{25}{4 \times 9} \\
&= \frac{25}{36}
\end{aligned}$$

Q53

$$\begin{aligned}
 & \lim_{x \rightarrow 0} \frac{\operatorname{cosec} x - \cot x}{x} \\
 &= \lim_{x \rightarrow 0} \left(\frac{1}{\sin x} - \frac{\cos x}{\sin x} \right) \times \frac{1}{x} \\
 &= \lim_{x \rightarrow 0} \left(\frac{1}{\sin x} \left(\frac{1 - \cos x}{x} \right) \right) \\
 &= \lim_{x \rightarrow 0} \left(\frac{1}{\sin x} \left(\frac{2 \sin^2 \frac{x}{2}}{x} \right) \right) \\
 &= 2 \lim_{x \rightarrow 0} \left(\frac{1}{\sin x} \times x \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}} \right)^2 \times \frac{x}{4} \right) \\
 &= 2 \left(\lim_{x \rightarrow 0} \frac{1}{\sin x} \right) \times \frac{1}{x} \times \left(\lim_{x \rightarrow 0} \frac{\sin \frac{x}{2}}{\frac{x}{2}} \right)^2 \times \frac{x}{4} \\
 &= 2 \times \frac{1}{x} \times \frac{x}{4} \\
 &= \frac{1}{2}
 \end{aligned}$$

Q54

$$\begin{aligned}
 & \lim_{x \rightarrow 0} \frac{\sin 3x + 7x}{4x + \sin 2x} \\
 &= \lim_{x \rightarrow 0} \frac{\left(\frac{\sin 3x}{x} + \frac{7x}{x} \right)}{\left(\frac{4x}{x} + \frac{\sin 2x}{x} \right)} \\
 &= \frac{\left(\lim_{x \rightarrow 0} \frac{\sin 3x}{3x} \times 3 \right) + 7}{4 + \left(\lim_{x \rightarrow 0} \frac{\sin 2x}{2x} \right) \times 2} \\
 &= \frac{3 + 7}{4 + 2} \\
 &= \frac{10}{6} \\
 &= \frac{5}{3}
 \end{aligned}$$

Q55

$$\begin{aligned}
& \lim_{x \rightarrow 0} \frac{5x + 4 \sin 3x}{4 \sin 2x + 7x} \\
&= \lim_{x \rightarrow 0} \frac{5 + \frac{4 \sin 3x}{x}}{\frac{4 \sin 2x}{x} + 7} \\
&= \frac{\lim_{x \rightarrow 0} 5 + 4 \lim_{x \rightarrow 0} \frac{\sin 3x}{3x} \times 3}{4 \lim_{x \rightarrow 0} \frac{\sin 2x}{2x} \times 2 + 7} \\
&= \frac{5 + 4 \times 1 \times 3}{4 \times 2 + 7} \\
&= \frac{5 + 12}{8 + 7} \\
&= \frac{17}{15}
\end{aligned}$$

Q56

$$\begin{aligned}
& \lim_{x \rightarrow 0} \frac{3 \sin x - \sin 3x}{x^3} \\
&= \lim_{x \rightarrow 0} \frac{3 \sin x - (3 \sin x - 4 \sin^3 x)}{x^3} \\
&= \lim_{x \rightarrow 0} \frac{4 \sin^3 x}{x^3} \\
&= 4 \left(\lim_{x \rightarrow 0} \frac{\sin x}{x} \right)^3 \\
&= 4 \times 1 \\
&= 4
\end{aligned}$$

Q57

$$\begin{aligned}
 & \lim_{x \rightarrow 0} \frac{\tan 2x - \sin 2x}{x^3} \\
 &= \lim_{x \rightarrow 0} \frac{\frac{\sin 2x}{\cos 2x} - \sin 2x}{x^3} \\
 &= \lim_{x \rightarrow 0} \frac{\sin 2x \left(\frac{1}{\cos 2x} - 1 \right)}{x^3} \\
 &= \lim_{x \rightarrow 0} \frac{\sin 2x (1 - \cos 2x)}{x^3 \cos 2x} \\
 &= \lim_{x \rightarrow 0} \frac{\sin 2x (2 \sin^2 x)}{x^3 \cos 2x} \\
 &= \frac{\left(\lim_{x \rightarrow 0} \frac{\sin 2x}{x} \right) \left(\lim_{x \rightarrow 0} \frac{2 \sin^2 x}{x^2} \right)}{\left(\lim_{x \rightarrow 0} \cos 2x \right)} \\
 &= \frac{\left(\lim_{x \rightarrow 0} \frac{\sin 2x}{2x} \times 2 \right) \left(\lim_{x \rightarrow 0} 2 \left(\frac{\sin x}{x} \right)^2 \right)}{\lim_{x \rightarrow 0} \cos 2x} \\
 &= \frac{(2 \times 1)(2 \times 1)}{1} \\
 &= 4
 \end{aligned}$$

Q58

$$\begin{aligned}
 & \lim_{x \rightarrow 0} \frac{\sin ax + bx}{ax + \sin bx} \\
 &= \lim_{x \rightarrow 0} \frac{\frac{\sin ax}{x} + b}{a + \frac{\sin bx}{x}} \\
 &= \frac{\lim_{x \rightarrow 0} \frac{\sin ax}{ax} \times a + b}{a + \lim_{x \rightarrow 0} \frac{\sin bx}{bx} \times b} \\
 &= \frac{a + b}{a + b} \\
 &= 1
 \end{aligned}$$

Q59

$$\begin{aligned}
 & \lim_{x \rightarrow 0} (\operatorname{cosec} x - \cot x) \\
 &= \lim_{x \rightarrow 0} \left(\frac{1}{\sin x} - \frac{\cos x}{\sin x} \right) \\
 &= \lim_{x \rightarrow 0} \left(\frac{1 - \cos x}{\sin x} \right) \\
 &= \lim_{x \rightarrow 0} \frac{2 \sin^2 \frac{x}{2}}{2 \sin \frac{x}{2} \cos \frac{x}{2}} \\
 &= \lim_{x \rightarrow 0} \frac{\tan x}{2} \\
 &= \left(\lim_{x \rightarrow 0} \frac{\tan x}{\frac{x}{2}} \right) \times \frac{x}{2} \\
 &= \lim_{x \rightarrow 0} 1 \times \frac{x}{2} \\
 &= 0
 \end{aligned}$$

Q60

$$\begin{aligned}
 & \lim_{x \rightarrow 0} \frac{\{\sin(\alpha + \beta)x + \sin(\alpha - \beta)x + \sin 2\alpha x\}}{\cos^2 \beta x - \cos^2 \alpha x} \\
 &= \lim_{x \rightarrow 0} \frac{\left\{ 2 \sin \frac{(\alpha + \beta + \alpha - \beta)}{2} x \cos \frac{(\alpha + \beta - \alpha + \beta)}{2} x + 2 \sin \alpha \cos \alpha x \right\}}{(\cos \beta x - \cos \alpha x)(\cos \beta x + \cos \alpha x)} \\
 &= \lim_{x \rightarrow 0} \frac{\{2 \sin \alpha x \cos \beta x + 2 \sin \alpha x \cos \alpha x\}}{(\cos \beta x - \cos \alpha x)(\cos \beta x + \cos \alpha x)} \\
 &= \lim_{x \rightarrow 0} \frac{2 \sin \alpha x (\cos \beta x + \cos \alpha x)}{(\cos \beta x - \cos \alpha x)(\cos \beta x + \cos \alpha x)} \\
 &= \lim_{x \rightarrow 0} \frac{2 \sin \alpha x}{(\cos \beta x - \cos \alpha x)} \\
 &= \lim_{x \rightarrow 0} \frac{2 \sin \alpha x}{\left(1 - 2 \sin^2 \left(\frac{\beta x}{2} \right) - 1 + 2 \sin^2 \left(\frac{\alpha x}{2} \right) \right)} \\
 &= \lim_{x \rightarrow 0} \frac{2 \sin \alpha x}{2 \sin^2 \left(\frac{\alpha x}{2} \right) - 2 \sin^2 \left(\frac{\beta x}{2} \right)} \\
 &= \frac{2\alpha}{\alpha^2 - \beta^2}
 \end{aligned}$$

Q61

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{\cos ax - \cos bx}{\cos cx - 1} &= \lim_{x \rightarrow 0} \frac{1 - 2\sin^2\left(\frac{ax}{2}\right) - 1 + 2\sin^2\left(\frac{bx}{2}\right)}{1 - 2\sin^2\left(\frac{cx}{2}\right) - 1} \\
 &= \lim_{x \rightarrow 0} \frac{-2\sin^2\left(\frac{ax}{2}\right) + 2\sin^2\left(\frac{bx}{2}\right)}{-2\sin^2\left(\frac{cx}{2}\right)} \\
 &= \lim_{x \rightarrow 0} \frac{-\sin^2\left(\frac{ax}{2}\right) 4a^2x^2 + \sin^2\left(\frac{bx}{2}\right) 4b^2x^2}{-\sin^2\left(\frac{cx}{2}\right) 4c^2x^2} \\
 &= \frac{-a^2 + b^2}{-c^2} \\
 &= \frac{a^2 - b^2}{c^2}
 \end{aligned}$$

Q62

$$\begin{aligned}
 \lim_{h \rightarrow 0} \frac{(a+h)^2 \sin(a+h) - a^2 \sin a}{h} &= \lim_{h \rightarrow 0} \frac{(a+h)^2 (\sin a \cosh + \cos a \sinh) - a^2 \sin a}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(a+h)^2 (\sin a \cosh) - a^2 \sin a + (a+h)^2 \cos a \sinh}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(a^2 + 2ah + h^2)(\sin a \cosh) - a^2 \sin a - (a+h)^2 \cos a \sinh}{h} \\
 &= \lim_{h \rightarrow 0} \frac{a^2 \sin a (\cosh - 1) + 2ah \sin a \cosh + h^2 \sin a \cosh + (a+h)^2 \cos a \sinh}{h} \\
 &= \lim_{h \rightarrow 0} \frac{a^2 \sin a (\cosh - 1)}{h} + \lim_{h \rightarrow 0} \frac{2ah \sin a \cosh}{h} - \lim_{h \rightarrow 0} \frac{a^2 \sin a \cosh}{h} + \lim_{h \rightarrow 0} \frac{(a+h)^2 \cos a \sinh}{h} \\
 &= \lim_{h \rightarrow 0} \frac{-a^2 \sin a \sin^2\left(\frac{h}{2}\right)}{\frac{h}{2}} + 2a \sin a + 0 + a^2 \cos a \\
 &= 0 + 2a \sin a + a^2 \cos a \\
 &= 2a \sin a + a^2 \cos a
 \end{aligned}$$

Q63

$$\lim_{x \rightarrow 0} kx \operatorname{cosec} x = \lim_{x \rightarrow 0} x \operatorname{cosec} kx$$

$$\lim_{x \rightarrow 0} kx \frac{1}{\sin x} = \lim_{x \rightarrow 0} x \frac{1}{\sin kx}$$

$$k \lim_{x \rightarrow 0} \left(\frac{x}{\sin x} \right) = \frac{1}{k} \lim_{x \rightarrow 0} \left(\frac{kx}{\sin kx} \right)$$

$$k = \frac{1}{k}$$

$$k^2 = 1$$

$$k = \pm 1$$

Ex 29.8

Q1

$$\lim_{x \rightarrow \pi/2} \left(\frac{\pi}{2} - x \right) \tan x$$

$$\text{Let } y = \frac{\pi}{2} - x$$

$$\text{as } x \rightarrow \pi/2, \quad y \rightarrow 0$$

$$\lim_{x \rightarrow \pi/2} \left(\frac{\pi}{2} - x \right) \tan x$$

$$= \lim_{y \rightarrow 0} y \tan \left(\frac{\pi}{2} - y \right)$$

$$= \lim_{y \rightarrow 0} y \frac{\sin \left(\frac{\pi}{2} - y \right)}{\cos \left(\frac{\pi}{2} - y \right)}$$

$$= \lim_{y \rightarrow 0} y \frac{\cos y}{\sin y}$$

$$= \lim_{y \rightarrow 0} \cos y = \lim_{y \rightarrow 0} \frac{y}{\sin y}$$

$$= 1$$

Q2

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{\sin 2x}{\cos x}$$

$$= \lim_{x \rightarrow \frac{\pi}{2}} \frac{2 \sin x \cos x}{\cos x}$$

$$= 2 \lim_{x \rightarrow \frac{\pi}{2}} \sin x$$

$$= 2 \times \sin \frac{\pi}{2}$$

$$= 2 \times 1$$

$$= 2$$

Q3

$$\begin{aligned} & \lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos^2 x}{1 - \sin x} \\ &= \lim_{x \rightarrow \frac{\pi}{2}} \frac{1 - \sin^2 x}{1 - \sin x} \\ &= \lim_{x \rightarrow \frac{\pi}{2}} \frac{(1 - \sin x)(1 + \sin x)}{(1 - \sin x)} \\ &= \lim_{x \rightarrow \frac{\pi}{2}} (1 + \sin x) \\ &= 1 + \sin \frac{\pi}{2} \\ &= 1 + 1 \\ &= 2 \end{aligned}$$

Q4

$$\begin{aligned} & \lim_{x \rightarrow \frac{\pi}{2}} \frac{1 - \sin x}{\cos^2 x} \\ &= \lim_{x \rightarrow \frac{\pi}{2}} \frac{1 - \sin x}{1 - \sin^2 x} \\ &= \lim_{x \rightarrow \frac{\pi}{2}} \frac{(1 - \sin x)}{(1 - \sin x)(1 + \sin x)} \\ &= \lim_{x \rightarrow \frac{\pi}{2}} \frac{1}{1 + \sin x} \\ &= \frac{1}{1 + \sin \frac{\pi}{2}} \\ &= \frac{1}{1 + 1} \\ &= \frac{1}{2} \end{aligned}$$

Q5

$$\lim_{x \rightarrow a} \frac{\cos x - \cos a}{x - a}$$

$$= \lim_{x \rightarrow a} \frac{\left(-2 \sin \left(\frac{x+a}{2} \right) \sin \left(\frac{x-a}{2} \right) \right)}{x - a}$$

$$= -2 \lim_{x \rightarrow a} \sin \left(\frac{x+a}{2} \right) \lim_{x \rightarrow a} \frac{\sin \left(\frac{x-a}{2} \right)}{x - a}$$

$$= -2 \times \sin \left(\frac{a+a}{2} \right) \times \left(\lim_{x \rightarrow a \rightarrow 0} \frac{\sin \left(\frac{x-a}{2} \right)}{\frac{x-a}{2}} \right) \times \frac{1}{2}$$

$$= -2 \sin a \times 1 \times \frac{1}{2}$$

$$= -\sin a$$

$$\left[\because \lim_{x \rightarrow a} \frac{\sin x}{x} = 1 \right]$$

Q6

$$\lim_{x \rightarrow \frac{\pi}{4}} \frac{1 - \tan x}{x - \frac{\pi}{4}}$$

$$\text{If } x \rightarrow \frac{\pi}{4}, \text{ then } x - \frac{\pi}{4} \rightarrow 0$$

$$\text{Let } x - \frac{\pi}{4} = y \Rightarrow y \rightarrow 0$$

$$= \lim_{y \rightarrow 0} \frac{1 - \tan \left(y + \frac{\pi}{4} \right)}{y}$$

$$= \lim_{y \rightarrow 0} \frac{1 - \left(\frac{\tan y + \tan \frac{\pi}{4}}{1 - \tan y \tan \frac{\pi}{4}} \right)}{y}$$

$$= \lim_{y \rightarrow 0} \frac{(1 - \tan y - \tan y - 1)}{y(1 - \tan y)}$$

$$= \lim_{y \rightarrow 0} \frac{(-2 \tan y)}{y(1 - \tan y)}$$

$$= -2 \lim_{y \rightarrow 0} \frac{\tan y}{y} \times \frac{1}{\lim_{y \rightarrow 0} (1 - \tan y)}$$

$$= -2 \times 1 \times \frac{1}{(1 - 0)}$$

$$= -2$$

$$\left[\because \tan \frac{\pi}{4} = 1 \right]$$

$$\left[\because \lim_{\theta \rightarrow 0} \frac{\tan \theta}{\theta} = 1 \right]$$

Q7

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{1 - \sin x}{\left(\frac{\pi}{2} - x\right)^2}$$

$$\text{If } x \rightarrow \frac{\pi}{2}, \frac{\pi}{2} - x \rightarrow 0$$

$$\text{Let } \frac{\pi}{2} - x = y \text{ then } y \rightarrow 0$$

$$= \lim_{y \rightarrow 0} \frac{1 - \sin\left(\frac{\pi}{2} - y\right)}{y^2}$$

$$= \lim_{y \rightarrow 0} \frac{1 - \cos y}{y^2}$$

$$= \lim_{y \rightarrow 0} \frac{2 \sin^2 \frac{y}{2}}{y^2}$$

$$= 2 \left(\lim_{y \rightarrow 0} \frac{\sin \frac{y}{2}}{\frac{y}{2}} \right)^2 \times \frac{1}{4}$$

$$= 2 \times 1 \times \frac{1}{4}$$

$$= \frac{1}{2}$$

$$\left[\because \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \right]$$

Q8

$$\lim_{x \rightarrow \frac{\pi}{3}} \frac{\sqrt{3} - \tan x}{\pi - 3x}$$

If $x \rightarrow \frac{\pi}{3}$, $\frac{\pi}{3} - x \rightarrow 0$, $\pi - 3x \rightarrow 0$

Let $\frac{\pi}{3} - x = y$ then $y \rightarrow 0$

$$\begin{aligned} &= \lim_{y \rightarrow 0} \frac{\sqrt{3} - \tan\left(\frac{\pi}{3} - y\right)}{3\left(\frac{\pi}{3} - x\right)} \\ &= \lim_{y \rightarrow 0} \left(\frac{\left(\sqrt{3} - \frac{\tan \frac{\pi}{3} - \tan y}{1 + \tan \frac{\pi}{3} \cdot \tan y} \right)}{3y} \right) \\ &= \lim_{y \rightarrow 0} \left(\frac{\left(\sqrt{3} - \frac{\sqrt{3} - \tan y}{1 + \sqrt{3} \tan y} \right)}{3y} \right) \\ &= \lim_{y \rightarrow 0} \frac{\left(\sqrt{3} + 3 \tan y - \sqrt{3} + \tan y \right)}{3(1 + \sqrt{3} \tan y)y} \\ &= \lim_{y \rightarrow 0} \frac{4 \tan y}{3(1 + \sqrt{3} \tan y)y} \\ &= \frac{4}{3} \times \lim_{y \rightarrow 0} \frac{\tan y}{y} \times \frac{1}{\lim_{y \rightarrow 0} \left(1 + \sqrt{3} \frac{\tan y}{y} \times y \right)} \\ &= \frac{4 \times 1}{3} \times \frac{1}{1 + 0} \\ &= \frac{4}{3} \end{aligned}$$

Q9

$$\lim_{x \rightarrow a} \frac{a \sin x - x \sin a}{ax^2 - xa^2} = \lim_{x \rightarrow a} \frac{a \sin x - x \sin a}{ax(x-a)}$$

Let $t = x - a$

Then, as $x \rightarrow a$, $t \rightarrow 0$

$$\begin{aligned} \therefore \lim_{x \rightarrow a} \frac{a \sin x - x \sin a}{ax(x-a)} &= \lim_{t \rightarrow 0} \frac{a \sin(t+a) - (t+a) \sin a}{a(t+a)t} \\ &= \lim_{t \rightarrow 0} \frac{a \sin t \cos a + a \sin a \cos t - t \sin a - a \sin a}{a(t+a)t} \\ &= \lim_{t \rightarrow 0} \frac{a \sin t \cos a + a \sin a (\cos t - 1) - t \sin a}{a(t+a)t} \\ &= \lim_{t \rightarrow 0} \frac{a \sin t \cos a + a \sin a (2 \sin^2(t/2)) - t \sin a}{a(t+a)t} \\ &= \lim_{t \rightarrow 0} \frac{a \sin t \cos a}{a(t+a)t} + \lim_{t \rightarrow 0} \frac{a \sin a (2 \sin^2(t/2))}{a(t+a)t} - \lim_{t \rightarrow 0} \frac{t \sin a}{a(t+a)t} \\ &= \frac{a \cos a}{a^2} + 0 - \frac{\sin a}{a^2} \\ &= \frac{a \cos a - \sin a}{a^2} \end{aligned}$$

Q10

$$\begin{aligned} & \lim_{x \rightarrow \frac{\pi}{2}} \frac{\sqrt{2} - \sqrt{1 + \sin x}}{\cos^2 x} \\ &= \lim_{x \rightarrow \frac{\pi}{2}} \frac{\sqrt{2} - \sqrt{1 + \sin x}}{\cos^2 x} \cdot \frac{\sqrt{2} + \sqrt{1 + \sin x}}{\sqrt{2} + \sqrt{1 + \sin x}} \\ &= \lim_{x \rightarrow \frac{\pi}{2}} \frac{2 - 1 - \sin x}{\cos^2 x (\sqrt{2} + \sqrt{1 + \sin x})} \\ &= \lim_{x \rightarrow \frac{\pi}{2}} \frac{1 - \sin x}{(1 - \sin^2 x) (\sqrt{2} + \sqrt{1 + \sin x})} \\ &= \lim_{x \rightarrow \frac{\pi}{2}} \frac{1}{(1 + \sin x) (\sqrt{2} + \sqrt{1 + \sin x})} \\ &= \frac{1}{(1 + 1) (\sqrt{2} + \sqrt{2})} \\ &= \frac{1}{(4\sqrt{2})} \end{aligned}$$

Q11

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{\sqrt{2 - \sin x} - 1}{\left(\frac{\pi}{2} - x\right)^2}$$

$$\Rightarrow x \rightarrow \frac{\pi}{2}, \text{ then } \frac{\pi}{2} - x \rightarrow 0, \text{ let } \frac{\pi}{2} - x = y$$

$$\lim_{x \rightarrow \frac{\pi}{2}} = \lim_{y \rightarrow 0} \frac{\sqrt{2 - \sin\left(\frac{\pi}{2} - y\right)} - 1}{y^2}$$

$$= \lim_{y \rightarrow 0} \frac{\sqrt{2 - \cos y} - 1}{y^2}$$

$$= \lim_{y \rightarrow 0} \frac{(\sqrt{2 - \cos y} - 1)}{y^2} \times \frac{(\sqrt{2 - \cos y} + 1)}{(\sqrt{2 - \cos y} + 1)}$$

$$= \lim_{y \rightarrow 0} \frac{(2 - \cos y - 1)}{y^2 (\sqrt{2 - \cos y} + 1)}$$

$$= \lim_{y \rightarrow 0} \frac{(1 - \cos y)}{(\sqrt{2 - \cos y} + 1) y^2}$$

$$= \lim_{y \rightarrow 0} \frac{2 \sin^2 \frac{y}{2}}{y^2 (\sqrt{2 - \cos y} + 1)}$$

$$= 2 \times \left(\lim_{y \rightarrow 0} \frac{\sin \frac{y}{2}}{\frac{y}{2}} \right)^2 \times \frac{1}{4} \times \frac{1}{\lim_{y \rightarrow 0} \sqrt{2 - \cos y} + 1}$$

$$= 2 \times 1 \times \frac{1}{4} \times \frac{1}{1+1} = \frac{1}{4}$$

Q12

$$\lim_{x \rightarrow \frac{\pi}{4}} \frac{\sqrt{2} - \cos x - \sin x}{\left(\frac{\pi}{4} - x\right)^2}$$

As $x \rightarrow \frac{\pi}{4}$, $\frac{\pi}{4} - x \rightarrow 0$, let $\frac{\pi}{4} - x = y$

$$\begin{aligned} &= \lim_{y \rightarrow 0} \frac{\sqrt{2} - \cos\left(\frac{\pi}{4} - y\right) - \sin\left(\frac{\pi}{4} - y\right)}{y^2} \\ &= \lim_{y \rightarrow 0} \frac{\sqrt{2} - \left[\cos\frac{\pi}{4}\cos y + \sin\frac{\pi}{4}\sin y + \sin\frac{\pi}{4}\cos y - \cos\frac{\pi}{4}\sin y\right]}{y^2} \\ &= \lim_{y \rightarrow 0} \frac{\sqrt{2} - \left(\frac{\cos y}{\sqrt{2}} + \frac{\sin y}{\sqrt{2}} + \frac{\cos y}{\sqrt{2}} - \frac{\sin y}{\sqrt{2}}\right)}{y^2} \\ &= \lim_{y \rightarrow 0} \frac{\sqrt{2} - \frac{2\cos y}{\sqrt{2}}}{y^2} = \lim_{y \rightarrow 0} \frac{\sqrt{2} - \sqrt{2}\cos y}{y^2} \\ &= \sqrt{2} \lim_{y \rightarrow 0} \frac{(1 - \cos y)}{y^2} \\ &= \sqrt{2} \lim_{y \rightarrow 0} \frac{2\sin^2 \frac{y}{2}}{\frac{y^2}{4}} \times \frac{1}{4} \\ &= \sqrt{2} \times 2 \times \frac{1}{4} \left(\lim_{y \rightarrow 0} \frac{\sin \frac{y}{2}}{\frac{y}{2}} \right)^2 \\ &= \sqrt{2} \times 2 \times \frac{1}{4} \times 1 \\ &= \frac{1}{\sqrt{2}} \end{aligned}$$

$$\left[\because \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \right]$$

Q13

$$\lim_{x \rightarrow \frac{\pi}{8}} \frac{\cot 4x - \cos 4x}{8^3 \left(\frac{\pi}{8} - x \right)^3}$$

When $x \rightarrow \frac{\pi}{8}$, $\frac{\pi}{8} - x \rightarrow 0$, let $\frac{\pi}{8} - x = y$

$$\begin{aligned} &= \lim_{y \rightarrow 0} \frac{\cot 4 \left(\frac{\pi}{8} - y \right) - \cos 4 \left(\frac{\pi}{8} - y \right)}{(8)^3 y^3} \\ &= \lim_{y \rightarrow 0} \frac{\cot \left(\frac{\pi}{2} - 4y \right) - \cos \left(\frac{\pi}{2} - 4y \right)}{(8)^3 y^3} \\ &= \lim_{y \rightarrow 0} \frac{\tan 4y - \sin 4y}{(8)^3 y^3} \\ &= \lim_{y \rightarrow 0} \frac{\frac{\sin 4y}{\cos 4y} - \sin 4y}{8^3 y^3} \\ &= \lim_{y \rightarrow 0} \frac{\sin 4y - \sin 4y \cos 4y}{\cos 4y \times y^3 \times 8^3} \\ &= \lim_{y \rightarrow 0} \frac{\sin 4y (1 - \cos 4y)}{\cos 4y \times y^3 \times 8^3} \\ &= \lim_{y \rightarrow 0} \frac{\sin 4y (2 \sin^2 2y)}{\cos 4y \times y^3 \times 8^3} \\ &= \frac{2}{8^3} \lim_{y \rightarrow 0} \frac{\sin 4y}{y} \times \frac{\sin^2 2y}{y^2} \times \frac{1}{\cos 4y} \\ &= \frac{2}{8^3} \left(\lim_{y \rightarrow 0} \frac{\sin 4y}{4y} \times 4 \right) \times \left(\lim_{y \rightarrow 0} \frac{\sin 2y}{2y} \right)^2 \times 4 \times \frac{1}{\lim_{y \rightarrow 0} \cos 4y} \\ &= \frac{2}{8^3} (1 \times 4) \times (1) \times 4 \times \frac{1}{1} \\ &= \frac{2 \times 4 \times 4}{8 \times 8 \times 8} \\ &= \frac{1}{16} \end{aligned}$$

$$\left[\because \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1, \lim_{\theta \rightarrow 0} \cos \theta = 1 \right]$$

Q14

$$\lim_{x \rightarrow a} \frac{\cos x - \cos a}{\sqrt{x} - \sqrt{a}}$$

$$\begin{aligned} &= \lim_{x \rightarrow a} \frac{\left(-2 \sin \left(\frac{x+a}{2} \right) \sin \left(\frac{x-a}{2} \right) \right) \times (\sqrt{x} + \sqrt{a})}{(\sqrt{x} - \sqrt{a})(\sqrt{x} + \sqrt{a})} \\ &= -2 \lim_{x \rightarrow a} \frac{\sin \left(\frac{x+a}{2} \right) \sin \left(\frac{x-a}{2} \right) \times (\sqrt{x} + \sqrt{a})}{(x-a)} \\ &= -2 \lim_{x \rightarrow a} \sin \left(\frac{x+a}{2} \right) \times \lim_{x \rightarrow a} \frac{\sin \left(\frac{x-a}{2} \right) \times \frac{1}{2}}{\left(\frac{x-a}{2} \right)} \times \lim_{x \rightarrow a} (\sqrt{x} + \sqrt{a}) \\ &= -2 \times \sin(a) \times 1 \times \frac{1}{2} \times 2\sqrt{a} \\ &= -2\sqrt{a} \sin a \end{aligned}$$

$$\left[\because \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \right]$$

Q15

$$\begin{aligned}
 & \lim_{x \rightarrow \pi} \frac{\sqrt{5 + \cos x} - 2}{(x - \pi)^2} \\
 \Rightarrow & \quad x \rightarrow \pi, \text{ then } x - \pi \rightarrow 0, \text{ let } x - \pi = y \\
 & = \lim_{y \rightarrow 0} \frac{\sqrt{5 + \cos(\pi - y)} - 2}{y^2} \\
 & = \lim_{y \rightarrow 0} \frac{\sqrt{5 - \cos y} - 2}{y^2} \\
 & = \lim_{y \rightarrow 0} \frac{(\sqrt{5 - \cos y} - 2)}{y^2} \times \frac{(\sqrt{5 - \cos y} + 2)}{(\sqrt{5 - \cos y} + 2)} \\
 & = \lim_{y \rightarrow 0} \frac{(5 - \cos y - 4)}{y^2 (\sqrt{5 - \cos y} + 2)} \\
 & = \lim_{y \rightarrow 0} \frac{2 \sin^2 \frac{y}{2}}{y^2 (\sqrt{5 - \cos y} + 2)} \\
 & = 2 \times \left(\lim_{y \rightarrow 0} \frac{\frac{\sin y}{2}}{\frac{y}{2}} \right)^2 \times \frac{1}{4 \lim_{y \rightarrow 0} (\sqrt{5 - \cos y} + 2)} \\
 & = 2 \times \frac{1}{4} \times \frac{1}{\sqrt{4} + 2} = 2 \times \frac{1}{4} \times \frac{1}{4} \\
 & = \frac{1}{8}
 \end{aligned}$$

Q16

$$\begin{aligned}
 & \lim_{x \rightarrow a} \frac{\cos \sqrt{x} - \cos \sqrt{a}}{x - a} \\
 & = \lim_{x \rightarrow a} \frac{-2 \sin \left(\frac{\sqrt{x} + \sqrt{a}}{2} \right) \sin \left(\frac{\sqrt{x} - \sqrt{a}}{2} \right)}{(\sqrt{x} - \sqrt{a})(\sqrt{x} + \sqrt{a})} \\
 & = -2 \lim_{x \rightarrow a} \frac{\sin \left(\frac{\sqrt{x} + \sqrt{a}}{2} \right) \times \lim_{x \rightarrow a} \sin \left(\frac{\sqrt{x} - \sqrt{a}}{2} \right)}{\lim_{x \rightarrow a} (\sqrt{x} + \sqrt{a}) \times \left(\frac{\sqrt{x} - \sqrt{a}}{2} \right)} \times \frac{1}{2} \\
 & = -2 \sin \sqrt{a} \times 1 \times \frac{1}{2\sqrt{a}} \times \frac{1}{2} \\
 & = -\frac{1}{2\sqrt{a}} \sin \sqrt{a}
 \end{aligned}$$

Q17

$$\begin{aligned}
 & \lim_{x \rightarrow a} \frac{\sin \sqrt{x} - \sin \sqrt{a}}{x - a} \\
 &= \lim_{x \rightarrow a} \frac{\sin \sqrt{x} - \sin \sqrt{a}}{(\sqrt{x} - \sqrt{a})(\sqrt{x} + \sqrt{a})} \\
 &= \lim_{x \rightarrow a} \frac{2 \sin \left(\frac{\sqrt{x} - \sqrt{a}}{2} \right) \cos \left(\frac{\sqrt{x} + \sqrt{a}}{2} \right)}{(\sqrt{x} - \sqrt{a})(\sqrt{x} + \sqrt{a})} \\
 &= 2 \left(\lim_{x \rightarrow a} \frac{\sin \frac{\sqrt{x} - \sqrt{a}}{2}}{\left(\frac{\sqrt{x} - \sqrt{a}}{2} \right)} \right) \times \frac{1}{2} \frac{\lim_{x \rightarrow a} \cos \left(\frac{\sqrt{x} + \sqrt{a}}{2} \right)}{\lim_{x \rightarrow a} (\sqrt{x} + \sqrt{a})} \\
 &= 2 \times 1 \times \frac{1}{2} \times \cos \sqrt{a} \times \frac{1}{2\sqrt{a}} \\
 &= \frac{\cos \sqrt{a}}{2\sqrt{a}}
 \end{aligned}$$

Q18

$$\begin{aligned}
 & \lim_{x \rightarrow 1} \frac{1 - x^2}{\sin 2\pi x} \\
 \Rightarrow & \quad x \rightarrow 1, \text{ then } x - 1 \rightarrow 0, \text{ let } x - 1 = y \\
 &= \lim_{(x-1) \rightarrow 0} \frac{(1-x)(1+x)}{\sin 2\pi x} \\
 &= \lim_{y \rightarrow 0} \frac{-y(1+y+1)}{\sin 2\pi(y+1)} \\
 &= - \lim_{y \rightarrow 0} \frac{y(y+2)}{\sin(2\pi y + 2\pi)} \\
 &= - \lim_{y \rightarrow 0} \frac{y(y+2)}{\sin 2\pi y} \\
 &= - \lim_{y \rightarrow 0} (y+2) \times \frac{y}{\left(\lim_{y \rightarrow 0} \sin \frac{2\pi y}{y \times 2\pi} \right) \times 2\pi y} \\
 &= -2 \times \frac{1}{1 \times 2\pi} \\
 &= -\frac{1}{\pi}
 \end{aligned}$$

Q19

$$\lim_{x \rightarrow \frac{\pi}{4}} \left(\frac{\sin 2x - \sin 2\left(\frac{\pi}{4}\right)}{x - \frac{\pi}{4}} \right)$$

[$\because$ given $f(x) = \sin 2x$]

$$= \lim_{x \rightarrow \frac{\pi}{4}} \left(\frac{\sin 2x - \sin \frac{\pi}{2}}{x - \frac{\pi}{4}} \right)$$

$$\Rightarrow x \rightarrow \frac{\pi}{4} \Rightarrow x - \frac{\pi}{4} \rightarrow 0, \text{ let } x - \frac{\pi}{4} = y$$

$$= \lim_{y \rightarrow 0} \left(\frac{\sin 2\left(y + \frac{\pi}{4}\right) - 1}{y} \right)$$

$$= \lim_{y \rightarrow 0} \frac{\sin\left(\frac{\pi}{2} + 2y\right) - 1}{y}$$

$$= \lim_{y \rightarrow 0} \frac{\cos 2y - 1}{y}$$

$$= - \lim_{y \rightarrow 0} \frac{1 - \cos 2y}{y}$$

$$= - \lim_{y \rightarrow 0} \frac{2 \sin^2 y}{y}$$

$$= -2 \left(\lim_{y \rightarrow 0} \frac{\sin y}{y} \right)^2 \times y$$

$$\left[\because \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \right]$$

$$= -2 \times 0$$

$$= 0$$

Q20

$$\lim_{x \rightarrow 1} \frac{1 + \cos \pi x}{(1 - x)^2}$$

$$\Rightarrow x \rightarrow 1, x - 1 \rightarrow 0, \text{ let } x - 1 = y$$

$$= \lim_{y \rightarrow 0} \frac{1 + \cos \pi(y + 1)}{(-y)^2}$$

$$= \lim_{y \rightarrow 0} \frac{1 + \cos(\pi + \pi y)}{y^2}$$

$$= \lim_{y \rightarrow 0} \frac{1 - \cos(\pi y)}{y^2}$$

$$= \lim_{y \rightarrow 0} \frac{2 \sin^2 \frac{\pi y}{2}}{y^2}$$

$$= 2 \left(\lim_{y \rightarrow 0} \frac{\sin \frac{\pi y}{2}}{\frac{\pi y}{2}} \right)^2 \times \frac{\pi^2}{4}$$

$$= 2 \times 1 \times \frac{\pi^2}{4}$$

$$= \frac{\pi^2}{2}$$

Q21

$$\lim_{x \rightarrow 1} \frac{1-x^2}{\sin \pi x}$$

$\Rightarrow x \rightarrow 1 \Rightarrow x-1 \rightarrow 0$, let $x-1=y \Rightarrow y \rightarrow 0$

$$\begin{aligned}\Rightarrow \lim_{x \rightarrow 1} \frac{1-x^2}{\sin \pi x} &= \lim_{x-1 \rightarrow 0} \frac{(1-x)(1+x)}{\sin \pi x} \\&= \lim_{y \rightarrow 0} \frac{(-y)(1+y+1)}{\sin \pi (y+1)} \\&= - \lim_{y \rightarrow 0} \frac{y(y+2)}{\sin (\pi y + \pi)} \\&= - \lim_{y \rightarrow 0} \frac{y(y+2)}{-\sin \pi y} \\&= \lim_{y \rightarrow 0} \frac{y(y+2)}{\sin \pi y} \\&= \frac{\lim_{y \rightarrow 0} y(y+2)}{\lim_{y \rightarrow 0} \frac{\sin \pi y}{\pi y} \times \pi y} \\&= \frac{2}{\pi}\end{aligned}$$

Q22

$$\lim_{x \rightarrow \frac{\pi}{4}} \frac{1 - \sin 2x}{1 + \cos 4x}$$

$$x \rightarrow \frac{\pi}{4}, x - \frac{\pi}{4} \rightarrow 0, \text{ let } x - \frac{\pi}{4} = y$$

$$\begin{aligned} \Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{1 - \sin 2x}{1 + \cos 4x} &= \lim_{x - \frac{\pi}{4} \rightarrow 0} \frac{\left(1 - \sin 2\left(y + \frac{\pi}{4}\right)\right)}{1 + \cos 4\left(y + \frac{\pi}{4}\right)} \\ &= \lim_{x - \frac{\pi}{4} \rightarrow 0} \left(\frac{1 - \sin\left(\frac{\pi}{2} + 2y\right)}{1 + \cos(\pi + 4y)} \right) \\ &= \lim_{x - \frac{\pi}{4} \rightarrow 0} \frac{1 - \cos 2y}{1 - \cos 4y} \\ &= \lim_{y \rightarrow 0} \frac{2 \sin^2 y}{2 \sin^2 2y} \\ &= \frac{\lim_{y \rightarrow 0} \sin^2 y}{\lim_{y \rightarrow 0} \sin^2 2y} \\ &= \frac{\left(\lim_{y \rightarrow 0} \frac{\sin y}{y}\right)^2 \times y^2}{\left(\lim_{y \rightarrow 0} \frac{\sin 2y}{2y}\right)^2 \times 4y^2} \\ &= \frac{1 \times y^2}{1 \times 4y^2} \quad \left[\because \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \right] \\ &= \frac{1}{4} \end{aligned}$$

Q23

$$\begin{aligned} \lim_{x \rightarrow \frac{\pi}{6}} \frac{\cot^2 x - 3}{\operatorname{cosec} x - 2} &= \lim_{x \rightarrow \frac{\pi}{6}} \frac{(\operatorname{cosec}^2 x - 1) - 3}{\operatorname{cosec} x - 2} \\ &= \lim_{x \rightarrow \frac{\pi}{6}} \frac{(\operatorname{cosec}^2 x - 4)}{\operatorname{cosec} x - 2} \\ &= \lim_{x \rightarrow \frac{\pi}{6}} \frac{(\operatorname{cosec} x - 2)(\operatorname{cosec} x + 2)}{(\operatorname{cosec} x - 2)} \\ &= \lim_{x \rightarrow \frac{\pi}{6}} \operatorname{cosec} x + 2 \\ &= \operatorname{cosec} \frac{\pi}{6} + 2 \\ &= 2 + 2 \\ &= 4 \end{aligned}$$

Q24

$$\begin{aligned} & \lim_{n \rightarrow \infty} n \sin\left(\frac{\pi}{4n}\right) \cos\left(\frac{\pi}{4n}\right) \\ &= \lim_{n \rightarrow \infty} 2 \left(n \sin \frac{\pi}{4n} \cos \frac{\pi}{4n} \right) \times \frac{1}{2} \\ &= \lim_{n \rightarrow \infty} n \times \sin \frac{\pi}{2n} \times \frac{1}{2} \end{aligned}$$

$$n \rightarrow \infty, \text{ then } \frac{1}{n} \rightarrow 0, \text{ let } \frac{1}{n} = y$$

$$= \frac{1}{2} \lim_{\frac{1}{n} \rightarrow 0} \frac{1}{y} \sin\left(\frac{\pi}{2}\right) \left(\frac{1}{n}\right)$$

$$= \frac{1}{2} \lim_{y \rightarrow 0} \frac{\sin\left(\frac{\pi}{2}\right) y}{y}$$

$$= \frac{1}{2} \left[\lim_{y \rightarrow 0} \frac{\sin\left(\frac{\pi y}{2}\right)}{\frac{\pi y}{2}} \right] \times \frac{\pi}{2}$$

$$= \frac{1}{2} \times 1 \times \frac{\pi}{2}$$

$$= \frac{\pi}{4}$$

$$\left[\because \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \right]$$

Q25

$$\lim_{n \rightarrow \infty} 2^{n-1} \sin\left(\frac{a}{2^n}\right)$$

$$= \lim_{n \rightarrow \infty} \frac{2^n}{2} \sin\left(\frac{a}{2^n}\right)$$

$$= \lim_{n \rightarrow \infty} \frac{2^n}{2} \sin \frac{a}{2^n}$$

$$n \rightarrow \infty, \frac{1}{n} \rightarrow 0, \text{ let } h = \frac{1}{n}$$

$$= \lim_{h \rightarrow 0} \frac{2^{\frac{1}{h}}}{2} \sin \frac{a}{2^{\frac{1}{h}}}$$

$$= \lim_{h \rightarrow 0} \frac{2^{\frac{1}{h}}}{2} \frac{\sin \frac{a}{2^{\frac{1}{h}}}}{\frac{a}{2^{\frac{1}{h}}}} \times \frac{a}{2^{\frac{1}{h}}}$$

$$= \frac{a}{2}$$

$$\left[\because \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \right]$$

Q26

$$\begin{aligned}
 &= \frac{\lim_{n \rightarrow \infty} \sin\left(\frac{a}{2^n}\right)}{\lim_{n \rightarrow \infty} \sin\left(\frac{b}{2^n}\right)} \\
 &n \rightarrow \infty, \frac{1}{n} = h \rightarrow 0 \\
 &= \frac{\lim_{h \rightarrow 0} \sin\left(\frac{a}{2^{\frac{1}{h}}}\right)}{\lim_{h \rightarrow 0} \sin\left(\frac{b}{2^{\frac{1}{h}}}\right)} \\
 &= \frac{\left(\lim_{h \rightarrow 0} \frac{\sin \frac{a}{2^{\frac{1}{h}}}}{\frac{a}{2^{\frac{1}{h}}}} \times \frac{a}{2^{\frac{1}{h}}} \right)}{\left(\lim_{h \rightarrow 0} \frac{\sin \frac{b}{2^{\frac{1}{h}}}}{\frac{b}{2^{\frac{1}{h}}}} \times \frac{b}{2^{\frac{1}{h}}} \right)} \\
 &= \frac{1 \times \frac{a}{2^{\frac{1}{h}}}}{1 \times \frac{b}{2^{\frac{1}{h}}}} = \frac{a}{b}
 \end{aligned}$$

Q27

$$\begin{aligned}
 &\lim_{x \rightarrow -1} \frac{x^2 - x - 2}{(x^2 + x) + \sin(x+1)} \\
 &= \lim_{x \rightarrow -1} \frac{(x-2)(x+1)}{(x+1) + \sin(x+1)} \\
 &= \lim_{x \rightarrow -1} \frac{1}{x(x+1) + \frac{\sin(x+1)}{(x-2)(x+1)}} \\
 &= \lim_{x \rightarrow -1} \frac{1}{\frac{x}{x-2} + \frac{\sin(x+1)}{(x-2)(x+1)}} \\
 &= \lim_{x \rightarrow -1} \frac{1}{(x-2)} \left(\frac{1}{x + \frac{\sin(x+1)}{x+1}} \right) \\
 &= \lim_{x \rightarrow -1} \frac{1}{x-2} \times \frac{1}{\lim_{x \rightarrow -1} (x) + \lim_{x+1 \rightarrow 0} \sin \frac{x+1}{x+1}} \\
 &= \left(\frac{1}{-1-2} \right) \times \frac{1}{(-1)+1} \\
 &= \frac{1}{0} \\
 &= \infty
 \end{aligned}$$

$$\begin{aligned}
 &\left[\because \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \right] \\
 &\left[\because \frac{1}{0} = \infty \right]
 \end{aligned}$$

Q28

$$\begin{aligned}
 & \lim_{x \rightarrow 2} \frac{x^2 - x - 2}{x^2 - 2x + \sin(x-2)} \\
 &= \lim_{x \rightarrow 2} \frac{(x-2)(x+1)}{x^2 - 2x + \sin(x-2)} \\
 &= \lim_{x \rightarrow 2} \frac{1}{\frac{x}{x+1} + \frac{\sin(x-2)}{(x-2)(x+1)}} \\
 &= \lim_{x \rightarrow 2} (x+1) \left(\frac{1}{x + \frac{\sin(x-2)}{x-2}} \right) \\
 &= \lim_{x \rightarrow 2} (x+1) \times \frac{1}{\lim_{x \rightarrow 2} (x) + \lim_{x \rightarrow 2} \frac{\sin(x-2)}{x-2}} \\
 &= (2+1) \times \frac{1}{(2) + \lim_{x \rightarrow 2} \frac{\sin(x-2)}{(x-2)}} \quad \left[\because \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \right] \\
 &= 3 \times \frac{1}{2+1} \\
 &= 1
 \end{aligned}$$

Q29

$$\lim_{x \rightarrow 1} (1-x) \tan \frac{\pi x}{2}$$

When $x \rightarrow 1$, $x-1 \rightarrow 0$, let $x-1 = y$, then $y \rightarrow 0$

$$\begin{aligned}
 &= \lim_{(x-1) \rightarrow 0} -(x-1) \tan \frac{\pi x}{2} \\
 &= - \lim_{y \rightarrow 0} y \tan \frac{\pi}{2} (y+1) \\
 &= - \lim_{y \rightarrow 0} y \times \tan \left(\frac{\pi}{2} + \frac{\pi}{2} y \right) \\
 &= \lim_{y \rightarrow 0} y \times \cot \frac{\pi}{2} y \\
 &= \lim_{y \rightarrow 0} \frac{y}{\tan \frac{\pi y}{2}} \\
 &= \lim_{y \rightarrow 0} \frac{\frac{\pi y}{2} \times \frac{2}{\pi}}{\tan \frac{\pi y}{2}} \\
 &= \frac{2}{\pi}
 \end{aligned}$$

Q30

$$\lim_{x \rightarrow \frac{\pi}{4}} \frac{1 - \tan x}{1 - \sqrt{2} \sin x}$$

$$x \rightarrow \frac{\pi}{4}, \text{ then } x - \frac{\pi}{4} \rightarrow 0, \text{ let } x - \frac{\pi}{4} = y$$

$$\begin{aligned} \Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{1 - \tan x}{1 - \sqrt{2} \sin x} &= \lim_{x - \frac{\pi}{4} \rightarrow 0} \frac{1 - \tan x}{1 - \sqrt{2} \sin x} \\ &= \lim_{y \rightarrow 0} \frac{1 - \tan\left(y + \frac{\pi}{4}\right)}{1 - \sqrt{2} \sin\left(y + \frac{\pi}{4}\right)} \\ &= \lim_{y \rightarrow 0} \frac{1 - \left(\frac{\tan \frac{\pi}{4} + \tan y}{1 + \tan \frac{\pi}{4} \tan y}\right)}{1 - \sqrt{2} \left(\sin y \cos \frac{\pi}{4} + \cos y \sin \frac{\pi}{4}\right)} \\ &= \lim_{y \rightarrow 0} \frac{\left(1 - \left(\frac{1 + \tan y}{1 - \tan y}\right)\right)}{1 - \sqrt{2} \left(\frac{\sin y}{\sqrt{2}} + \frac{\cos y}{\sqrt{2}}\right)} \\ &= \lim_{y \rightarrow 0} \frac{(1 - \tan y - 1 - \tan y)}{(1 - \tan y)(1 - \sin y + \cos y)} \\ &= \lim_{y \rightarrow 0} \left(\frac{-2 \tan y}{(1 - \tan y)(1 - \sin y - \cos y)}\right) \\ &= -2 \lim_{y \rightarrow 0} \frac{\tan y \times 1}{\lim_{y \rightarrow 0} (1 - \tan y) \times \lim_{y \rightarrow 0} (1 - \sin y - \cos y)} \\ &= \frac{-2 \left(\lim_{y \rightarrow 0} \frac{\tan y}{y}\right) \times y}{\left(\lim_{y \rightarrow 0} (1) - \lim_{y \rightarrow 0} \tan y\right) \times \left(1 - \lim_{y \rightarrow 0} \frac{\sin y}{y} \times y - \cos 0\right)} \\ &= \frac{-2}{(1 - y)(1 - y - 1)} = \frac{-2y}{(1 - y)(-y)} = \frac{2}{1 - y} \\ &= \lim_{y \rightarrow 0} \frac{2}{1 - y} = 2 \\ &= 2 \end{aligned}$$

Q31

$$\begin{aligned}
 & \lim_{x \rightarrow \pi} \frac{\sqrt{2 + \cos x} - 1}{(x - \pi)^2} \\
 \Rightarrow & \quad x \rightarrow \pi \text{ then } x - \pi \rightarrow 0 \text{ or let } x - \pi = y \\
 \Rightarrow & \quad \lim_{x \rightarrow \pi} \frac{\sqrt{2 + \cos x} - 1}{(x - \pi)^2} = \lim_{x - \pi \rightarrow 0} \frac{\sqrt{2 + \cos(x)} - 1}{(-1)^2 (x - \pi)^2} \\
 & = \lim_{y \rightarrow 0} \frac{\sqrt{2 + \cos(\pi + y)} - 1}{y^2} \\
 & = \lim_{y \rightarrow 0} \frac{\sqrt{2 - \cos y} - 1}{y^2} \\
 & = \lim_{y \rightarrow 0} \frac{(\sqrt{2 - \cos y} - 1)(\sqrt{2 - \cos y} + 1)}{y^2 (\sqrt{2 - \cos y} + 1)} \\
 & = \lim_{y \rightarrow 0} \frac{(2 - \cos y - 1)}{(\sqrt{2 - \cos y} + 1) y^2} \\
 & = \lim_{y \rightarrow 0} \frac{(1 - \cos y)}{(\sqrt{2 - \cos y} + 1) y^2} \\
 & = \lim_{y \rightarrow 0} \frac{2 \sin^2 \frac{y}{2}}{y^2 (\sqrt{2 - \cos y} + 1)} \\
 & = 2 \lim_{y \rightarrow 0} \left(\frac{\frac{\sin y}{2}}{\frac{y}{2}} \right)^2 \times \frac{1}{4} \times \frac{1}{\lim_{y \rightarrow 0} \sqrt{2 - \cos 0} + 1} \\
 & = 2 \times \frac{1}{4} \times \frac{1}{\sqrt{2 - 1} + 1} \\
 & = \frac{1}{4}
 \end{aligned}$$

$$\left[\because \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \right]$$

Q32

$$\begin{aligned} & \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sqrt{\cos x} - \sqrt{\sin x}}{x - \frac{\pi}{4}} \\ &= \lim_{x \rightarrow \frac{\pi}{4}} \frac{(\sqrt{\cos x} - \sqrt{\sin x})}{\left(x - \frac{\pi}{4}\right)} \times \frac{(\sqrt{\cos x} + \sqrt{\sin x})}{(\sqrt{\cos x} + \sqrt{\sin x})} \\ &= \lim_{x \rightarrow \frac{\pi}{4}} \frac{(\cos x - \sin x)}{\left(x - \frac{\pi}{4}\right)(\sqrt{\cos x} + \sqrt{\sin x})} \end{aligned}$$

$$\text{As } x \rightarrow \frac{\pi}{4} \Rightarrow x - \frac{\pi}{4} \rightarrow 0 \Rightarrow \text{let } x - \frac{\pi}{4} = y$$

$$\begin{aligned} &= \lim_{y \rightarrow 0} \frac{\left(\cos\left(\frac{\pi}{4} + y\right) - \sin\left(\frac{\pi}{4} + y\right)\right)}{y \left(\sqrt{\cos\left(\frac{\pi}{4} + y\right)} + \sqrt{\sin\left(\frac{\pi}{4} + y\right)}\right)} \\ &= \lim_{y \rightarrow 0} \frac{\left(\cos \frac{\pi}{4} \cos y - \sin \frac{\pi}{4} \sin y\right) - \left(\sin \frac{\pi}{4} \cos y + \cos \frac{\pi}{4} \sin y\right)}{y \left(\sqrt{\cos\left(\frac{\pi}{4} + y\right)} + \sqrt{\sin\left(\frac{\pi}{4} + y\right)}\right)} \\ &= \lim_{y \rightarrow 0} \frac{\left(\frac{\cos y}{\sqrt{2}} - \frac{\sin y}{\sqrt{2}} - \frac{\cos y}{\sqrt{2}} - \frac{\sin y}{\sqrt{2}}\right)}{y \left(\sqrt{\cos\left(\frac{\pi}{4} + y\right)} + \sqrt{\sin\left(\frac{\pi}{4} + y\right)}\right)} \\ &= \lim_{y \rightarrow 0} \frac{\left(-2 \frac{\sin y}{\sqrt{2}}\right)}{y \left(\sqrt{\cos\left(y + \frac{\pi}{4}\right)} + \sqrt{\sin\left(y + \frac{\pi}{4}\right)}\right)} \\ &= -\sqrt{2} \left(\lim_{y \rightarrow 0} \frac{\sin y}{y}\right) \times \frac{1}{\lim_{y \rightarrow 0} \sqrt{\cos\left(y + \frac{\pi}{4}\right)} + \lim_{y \rightarrow 0} \sqrt{\sin\left(y + \frac{\pi}{4}\right)}} \\ &= -\sqrt{2} \times 1 \times \frac{1}{\sqrt{\cos \frac{\pi}{4}} + \sqrt{\sin \frac{\pi}{4}}} \\ &= -\sqrt{2} \times \frac{1}{\left(\frac{1}{\sqrt{2}}\right)^{\frac{1}{2}} + \left(\frac{1}{\sqrt{2}}\right)^{\frac{1}{2}}} \end{aligned}$$

$$\left[\because \cos \frac{\pi}{4} = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}} \right]$$

$$\begin{aligned} &= \frac{-\sqrt{2}}{\left(\frac{1}{\sqrt{2}}\right)^{\frac{1}{2}} + \left(\frac{1}{\sqrt{2}}\right)^{\frac{1}{2}}} \\ &= \frac{-\sqrt{2}}{\sqrt{2} \left(\frac{1}{\sqrt{2}}\right)^{\frac{1}{2}}} \\ &= -\frac{1}{2^{\frac{1}{4}}} \end{aligned}$$

Q33

$$\lim_{x \rightarrow 1} \frac{1 - \frac{1}{x}}{\sin \pi (x-1)}$$

As $x \rightarrow 1$, then $x-1 \rightarrow 0$ let $x-1 = y$

$$\begin{aligned} &= \lim_{x \rightarrow 1} \frac{(x-1)}{x \times \sin \pi (x-1)} \\ &= \lim_{y \rightarrow 0} \frac{y}{(y+1) \sin(\pi y)} \\ &= \lim_{y \rightarrow 0} \frac{y}{(y+1) (\sin \pi y)} \\ &= \lim_{y \rightarrow 0} \frac{1}{(y+1) \sin \pi y} \\ &= \frac{1}{\left(\lim_{y \rightarrow 0} (y+1) \right) \times \left(\lim_{y \rightarrow 0} \frac{\sin \pi y}{y \times \pi} \times \pi \right)} \\ &= \frac{1}{(1)(1 \times \pi)} \\ &= \frac{1}{\pi} \end{aligned}$$

$$\left[\because \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \right]$$

Q34

$$\lim_{x \rightarrow \frac{\pi}{4}} \frac{\sqrt{2} - \cos x - \sin x}{(4x - \pi)^2}$$

$$x \rightarrow \frac{\pi}{4} \text{ then } x - \frac{\pi}{4} \rightarrow 0, \text{ also } 4x - \pi \rightarrow 0 \text{ let } x - \frac{\pi}{4} \rightarrow y$$

$$\lim_{x - \frac{\pi}{4} \rightarrow 0} \frac{\sqrt{2} - \cos x - \sin x}{(4)^2 \left(x - \frac{\pi}{4}\right)^2}$$

$$= \lim_{y \rightarrow 0} \frac{\sqrt{2} - \cos\left(y + \frac{\pi}{4}\right) - \sin\left(y + \frac{\pi}{4}\right)}{16 \times y^2}$$

$$= \frac{1}{16} \lim_{y \rightarrow 0} \frac{\sqrt{2} - \left(\cos y \cos \frac{\pi}{4} - \sin y \sin \frac{\pi}{4}\right) - \left(\sin y \cos \frac{\pi}{4} + \cos y \sin \frac{\pi}{4}\right)}{y^2}$$

$$= \frac{1}{16} \lim_{y \rightarrow 0} \frac{\sqrt{2} - \left(\cos y \times \frac{1}{\sqrt{2}} - \sin y \times \frac{1}{\sqrt{2}}\right) - \left(\frac{\sin y}{\sqrt{2}} + \frac{\cos y}{\sqrt{2}}\right)}{y^2}$$

$$= \frac{1}{16} \lim_{y \rightarrow 0} \frac{\sqrt{2} - \frac{1}{\sqrt{2}}(\cos y - \sin y) - \frac{1}{\sqrt{2}}(\sin y + \cos y)}{y^2}$$

$$= \frac{1}{16} \lim_{y \rightarrow 0} \frac{\sqrt{2} - \frac{1}{\sqrt{2}}[(\cos y - \sin y) + (\sin y + \cos y)]}{y^2}$$

$$= \frac{1}{16} \lim_{y \rightarrow 0} \frac{\left(\sqrt{2} - \frac{1}{\sqrt{2}}\cos y + \frac{1}{\sqrt{2}}\sin y - \frac{1}{\sqrt{2}}\sin y - \frac{1}{\sqrt{2}}\cos y\right)}{y^2}$$

$$= \frac{1}{16} \lim_{y \rightarrow 0} \frac{\sqrt{2} + \frac{2}{\sqrt{2}}\cos y}{y^2}$$

$$= \frac{1}{16} \lim_{y \rightarrow 0} \frac{\sqrt{2}(1 + \cos y)}{y^2}$$

$$= \frac{\sqrt{2}}{16} \lim_{y \rightarrow 0} \frac{2 \sin^2 \frac{y}{2}}{y^2}$$

$$= \frac{\sqrt{2}}{8} \lim_{y \rightarrow 0} \left(\frac{\sin \frac{y}{2}}{\frac{y}{2}}\right)^2 \times \frac{1}{4}$$

$$= \frac{1}{16\sqrt{2}}$$

Q35

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{\left(\frac{\pi}{2} - x\right) \sin x - 2 \cos x}{\left(\frac{\pi}{2} - x\right) + \cot x}$$

$$\text{If } x \rightarrow \frac{\pi}{2}, \text{ then } \frac{\pi}{2} - x \rightarrow 0$$

$$\text{Let } \frac{\pi}{2} - x = y.$$

$$= \lim_{y \rightarrow 0} \frac{\left(y \sin\left(\frac{\pi}{2} - y\right) - 2 \cos\left(\frac{\pi}{2} - y\right)\right)}{y + \cot\left(\frac{\pi}{2} - y\right)}$$

$$= \lim_{y \rightarrow 0} \left(\frac{y \cos y - 2 \sin y}{1 + \tan y} \right)$$

$$= \lim_{y \rightarrow 0} \left(\frac{\cos y - 2 \frac{\sin y}{y}}{1 + \frac{\tan y}{y}} \right)$$

$$= \frac{\lim_{y \rightarrow 0} \cos y - 2 \lim_{y \rightarrow 0} \frac{\sin y}{y}}{1 + \lim_{y \rightarrow 0} \frac{\tan y}{y}}$$

$$= \frac{1 - 2}{1 + 1} = \frac{-1}{2}$$

$$= -\frac{1}{2}$$

$$\left[\because \lim_{y \rightarrow 0} \frac{\sin y}{y} = 1, \lim_{y \rightarrow 0} \frac{\tan y}{y} = 1 \right]$$

Q36

$$\lim_{x \rightarrow \frac{\pi}{4}} \frac{\cos x - \sin x}{\left(\frac{\pi}{4} - x\right)(\cos x + \sin x)}$$

$$\Rightarrow x \rightarrow \frac{\pi}{4} \text{ then } x - \frac{\pi}{4} \rightarrow 0, \text{ let } x - \frac{\pi}{4} = y$$

$$\begin{aligned} &= \lim_{y \rightarrow 0} \frac{\cos\left(\frac{\pi}{4} + y\right) - \sin\left(\frac{\pi}{4} + y\right)}{-y \left(\cos\left(\frac{\pi}{4} + y\right) + \sin\left(\frac{\pi}{4} + y\right)\right)} \\ &= \lim_{y \rightarrow 0} \frac{\left[\left(\cos \frac{\pi}{4} \cos y - \sin \frac{\pi}{4} \sin y\right) - \left(\sin \frac{\pi}{4} \cos y + \cos \frac{\pi}{4} \sin y\right)\right]}{-y \left(\cos\left(\frac{\pi}{4} + y\right) + \sin\left(\frac{\pi}{4} + y\right)\right)} \\ &= \lim_{y \rightarrow 0} \frac{\left[\frac{\cos y}{\sqrt{2}} - \frac{\sin y}{\sqrt{2}} - \frac{\cos y}{\sqrt{2}} - \frac{\sin y}{\sqrt{2}}\right]}{-y \left(\cos\left(\frac{\pi}{4} + y\right) + \sin\left(\frac{\pi}{4} + y\right)\right)} \\ &= \lim_{y \rightarrow 0} \frac{\frac{-2 \sin y}{\sqrt{2}}}{-y \left(\cos\left(\frac{\pi}{4} + y\right) + \sin\left(\frac{\pi}{4} + y\right)\right)} \\ &= \sqrt{2} \lim_{y \rightarrow 0} \left(\frac{\sin y}{y}\right) \times \frac{1}{\lim_{y \rightarrow 0} \left(\cos\left(\frac{\pi}{4} + y\right) + \sin\left(\frac{\pi}{4} + y\right)\right)} \\ &= \sqrt{2} \times 1 \times \frac{1}{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}} = \sqrt{2} \times \frac{1}{\frac{2}{\sqrt{2}}} \\ &= \frac{\sqrt{2} \times \sqrt{2}}{2} = 1 \end{aligned}$$

Q37

$$\begin{aligned}\lim_{x \rightarrow \pi} \frac{1 - \sin \frac{x}{2}}{\cos \frac{x}{2} \left(\cos \frac{x}{4} - \sin \frac{x}{4} \right)} &= \lim_{h \rightarrow 0} \frac{1 - \sin \left(\frac{\pi+h}{2} \right)}{\cos \left(\frac{\pi+h}{2} \right) \left(\cos \left(\frac{\pi+h}{4} \right) - \sin \left(\frac{\pi+h}{4} \right) \right)} \\&= \lim_{h \rightarrow 0} \frac{1 - \cos \left(\frac{h}{2} \right)}{-\sin \left(\frac{h}{2} \right) \left(\frac{1}{\sqrt{2}} \cos \left(\frac{h}{4} \right) - \frac{1}{\sqrt{2}} \sin \left(\frac{h}{4} \right) - \frac{1}{\sqrt{2}} \sin \left(\frac{h}{4} \right) - \frac{1}{\sqrt{2}} \cos \left(\frac{h}{4} \right) \right)} \\&= \lim_{h \rightarrow 0} \frac{1 - \cos \left(\frac{h}{2} \right)}{\sqrt{2} \sin \left(\frac{h}{2} \right) \sin \left(\frac{h}{4} \right)} \\&= \lim_{h \rightarrow 0} \frac{2 \sin^2 \left(\frac{h}{4} \right)}{\sqrt{2} \sin \left(\frac{h}{2} \right) \sin \left(\frac{h}{4} \right)} \\&= \sqrt{2} \lim_{h \rightarrow 0} \frac{\sin \left(\frac{h}{4} \right)}{\sin \left(\frac{h}{2} \right)}\end{aligned}$$

Ex 29.9

Q1

$$= \lim_{x \rightarrow \pi} \frac{1 + \cos x}{\tan^2 x}$$

As $x \rightarrow \pi$, $x - \pi \rightarrow 0$, let $x - \pi = y$

$$= \lim_{y \rightarrow 0} \frac{1 + \cos(\pi + y)}{\tan^2(\pi + y)}$$

$$= \lim_{y \rightarrow 0} \frac{1 - \cos y}{\tan^2 y}$$

$$= \lim_{y \rightarrow 0} \frac{2 \sin^2 \frac{y}{2}}{\tan^2 y}$$

$$= \frac{\lim_{y \rightarrow 0} 2 \sin^2 \frac{y}{2}}{\lim_{y \rightarrow 0} \tan^2 y}$$

$$= \frac{2 \left(\lim_{y \rightarrow 0} \frac{\sin y}{\frac{y}{2}} \right)^2 \times \frac{y^2}{4}}{\left(\lim_{y \rightarrow 0} \frac{\tan y}{y} \right) \times y^2}$$

$$= \frac{2 \times 1 \times \frac{y^2}{4}}{1 \times y^2}$$

$$= 2 \times 1 \times \frac{1}{4}$$

$$= \frac{1}{2}$$

$$\left[\begin{array}{l} \because \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \\ \lim_{\theta \rightarrow 0} \frac{\tan \theta}{\theta} = 1 \end{array} \right]$$

Q2

$$\begin{aligned} & \lim_{x \rightarrow \frac{\pi}{4}} \frac{\operatorname{cosec}^2 x - 2}{\cot x - 1} \\ &= \lim_{x \rightarrow \frac{\pi}{4}} \frac{\cot^2 x + 1 - 2}{\cot x - 1} \\ &= \lim_{x \rightarrow \frac{\pi}{4}} \frac{\cot^2 x - 1}{\cot x - 1} \\ &= \lim_{x \rightarrow \frac{\pi}{4}} \frac{(\cot x - 1)(\cot x + 1)}{(\cot x - 1)} \\ &= \lim_{x \rightarrow \frac{\pi}{4}} (\cot x + 1) \\ &= \cot \frac{\pi}{4} + 1 \\ &= 1 + 1 \\ &= 2 \end{aligned}$$

Q3

$$\begin{aligned} & \lim_{x \rightarrow \frac{\pi}{6}} \frac{\operatorname{cosec}^2 x - 1 - 3}{\operatorname{cosec} x - 2} \\ &= \lim_{x \rightarrow \frac{\pi}{6}} \frac{\operatorname{cosec}^2 x - 4}{\operatorname{cosec} x - 2} \\ &= \lim_{x \rightarrow \frac{\pi}{6}} \frac{(\operatorname{cosec} x - 2)(\operatorname{cosec} x + 2)}{(\operatorname{cosec} x - 2)} \\ &= \lim_{x \rightarrow \frac{\pi}{6}} (\operatorname{cosec} x + 2) \\ &= \operatorname{cosec} \frac{\pi}{6} + 2 \\ &= 2 + 2 \\ &= 4 \end{aligned}$$

Q4

$$\begin{aligned} & \lim_{x \rightarrow \frac{\pi}{4}} \frac{2 - \operatorname{cosec}^2 x}{1 - \cot x} \\ &= \lim_{x \rightarrow \frac{\pi}{4}} \frac{2 - (1 + \cot^2 x)}{1 - \cot x} \\ &= \lim_{x \rightarrow \frac{\pi}{4}} \frac{2 - 1 - \cot^2 x}{1 - \cot x} \\ &= \lim_{x \rightarrow \frac{\pi}{4}} \frac{1 - \cot^2 x}{1 - \cot x} \\ &= \lim_{x \rightarrow \frac{\pi}{4}} \frac{(1 - \cot x)(1 + \cot x)}{(1 - \cot x)} \\ &= \lim_{x \rightarrow \frac{\pi}{4}} (1 + \cot x) \\ &= 1 + \cot \frac{\pi}{4} \\ &= 1 + 1 \\ &= 2 \end{aligned}$$

Q5

$$\begin{aligned}
 & \lim_{x \rightarrow \pi} \frac{\sqrt{2 + \cos x} - 1}{(\pi - x)^2} \\
 &= \lim_{x \rightarrow \pi} \frac{\sqrt{2 + \cos x} - 1}{(\pi - x)^2} \times \frac{\sqrt{2 + \cos x} + 1}{\sqrt{2 + \cos x} + 1} \\
 &= \lim_{x \rightarrow \pi} \frac{(2 + \cos x) - 1}{(\pi - x)^2 (\sqrt{2 + \cos x} + 1)} \\
 &= \lim_{x \rightarrow \pi} \frac{1 + \cos x}{(\pi - x)^2 (\sqrt{2 + \cos x} + 1)}
 \end{aligned}$$

Let $\pi - x = y, x \rightarrow \pi, y \rightarrow 0$

$$\Rightarrow \lim_{x \rightarrow \pi} \frac{1 + \cos x}{(\pi - x)^2 (\sqrt{2 + \cos x} + 1)} = \lim_{y \rightarrow 0} \frac{1 + \cos(\pi - y)}{y^2 (\sqrt{2 + \cos(\pi - y)} + 1)}$$

$$\begin{aligned}
 &= \lim_{y \rightarrow 0} \frac{1 - \cos y}{y^2 \sqrt{2 - \cos y} + 1} \\
 &= \lim_{y \rightarrow 0} \frac{2 \sin^2 y}{y^2 \sqrt{2 - \cos y} + 1} \\
 &= \lim_{y \rightarrow 0} \frac{2}{y^2 \sqrt{2 - \cos y} + 1} \\
 &= 2 \lim_{y \rightarrow 0} \left(\frac{\sin y}{\frac{y}{2}} \right)^2 \times \frac{1}{4 \sqrt{2 - \cos y} + 1} \\
 &= 2 \times \left(\lim_{y \rightarrow 0} \frac{\sin y}{\frac{y}{2}} \right)^2 \times \frac{1}{4 \lim_{y \rightarrow 0} \sqrt{2 - \cos y} + 1} \\
 &= 2 \times 1 \times \frac{1}{4} \times \frac{1}{\sqrt{2 - \cos 0} + 1} \\
 &= 2 \times 1 \times \frac{1}{4} \times \frac{1}{\sqrt{2 - 1} + 1} \\
 &= 2 \times 1 \times \frac{1}{4} \times \frac{1}{1 + 1} \\
 &= 2 \times 1 \times \frac{1}{4} \times \frac{1}{2} \\
 &= \frac{1}{4}
 \end{aligned}$$

Q6

$$\begin{aligned}& \lim_{x \rightarrow \frac{3\pi}{2}} \frac{1 + \operatorname{cosec}^3 x}{\cot^2 x} \\&= \lim_{x \rightarrow \frac{3\pi}{2}} \frac{(1 + \operatorname{cosec} x)(1 + \operatorname{cosec}^2 x - \operatorname{cosec} x)}{(\operatorname{cosec}^2 x - 1)} \\&= \lim_{x \rightarrow \frac{3\pi}{2}} \frac{(\operatorname{cosec} x + 1)(1 + \operatorname{cosec}^2 x - \operatorname{cosec} x)}{(\operatorname{cosec} x - 1)(\operatorname{cosec} x + 1)} \\&= \lim_{x \rightarrow \frac{3\pi}{2}} \frac{(1 + \operatorname{cosec}^2 x - \operatorname{cosec} x)}{\operatorname{cosec} x - 1} \\&= \frac{1 + \operatorname{cosec}^2 \frac{3\pi}{2} - \operatorname{cosec} \frac{3\pi}{2}}{\operatorname{cosec} \frac{3\pi}{2} - 1} \\&= \frac{1 + (-1)^2 - (-1)}{(-1) - 1} \\&= \frac{1 + 1 + 1}{-2} \\&= \frac{-3}{2}\end{aligned}$$

$$\left[\because \operatorname{cosec} \frac{3\pi}{2} = -1 \right]$$

Ex 29.10

Q1

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{5^x - 1}{\sqrt{4+x} - 2} \\&= \lim_{x \rightarrow 0} \frac{(5^x - 1)(\sqrt{4+x} + 2)}{(\sqrt{4+x} - 2)(\sqrt{4+x} + 2)} \\&= \lim_{x \rightarrow 0} \frac{(5^x - 1)(\sqrt{4+x} + 2)}{x} \\&= 4 \log 5\end{aligned}$$

Q2

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\log(1+x)}{3^x - 1} \\&= \lim_{x \rightarrow 0} \frac{\log(1+x)}{x} \times \frac{1}{\lim_{x \rightarrow 0} \frac{3^x - 1}{x}} \\&= \frac{1}{\log 3}\end{aligned}$$

Q3

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{a^x + a^{-x} - 2}{x^2} \\&= \lim_{x \rightarrow 0} \frac{a^{2x} - 2a^x + 1}{a^x \cdot x^2} \\&= \lim_{x \rightarrow 0} \left(\frac{a^x - 1}{x} \right)^2 \times \frac{1}{a^x} \\&= (\log_e a)^2 \times \frac{1}{a^0} \\&= (\log_e a)^2\end{aligned}$$

Q4

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{a^{mx} - 1}{b^{nx} - 1}, n \neq 0 \\ &= \lim_{x \rightarrow 0} \frac{a^{mx} - 1}{mx} \times \frac{1}{\lim_{x \rightarrow 0} \frac{b^{nx} - 1}{nx}} \times \frac{m}{n} \\ &= \frac{m \log a}{n \log b}, n \neq 0 \end{aligned}$$

Q5

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{a^x + b^x - 2}{x} \\ &= \lim_{x \rightarrow 0} \frac{a^x - 1}{x} + \lim_{x \rightarrow 0} \frac{b^x - 1}{x} \\ &= \log a + \log b \\ &= \log(ab) \end{aligned}$$

Q6

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{9^x - 2 \cdot 6^x + 4^x}{x^2} \\ &= \lim_{x \rightarrow 0} \frac{(3^x)^2 - 2 \cdot 3^x 2^x + (2^x)^2}{x^2} \\ &= \lim_{x \rightarrow 0} \left(\frac{3^x - 2^x}{x} \right)^2 \\ &= \left(\lim_{x \rightarrow 0} \frac{3^x - 1}{x} - \lim_{x \rightarrow 0} \left(\frac{2^x - 1}{x} \right) \right)^2 \\ &= \left(\log \frac{3}{2} \right)^2 \end{aligned}$$

Q7

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{8^x - 4^x - 2^x + 1}{x^2} \\&= \lim_{x \rightarrow 0} \frac{(2^x - 1)^2 (2^x + 1)}{x^2} \\&= \lim_{x \rightarrow 0} \left(\frac{(2^x - 1)}{x} \right)^2 \lim_{x \rightarrow 0} (2^x + 1) \\&= 2 (\log 2)^2\end{aligned}$$

Q8

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{a^{mx} - b^{nx}}{x} \\&= \lim_{x \rightarrow 0} m \frac{a^{mx} - 1}{mx} - \lim_{x \rightarrow 0} n \frac{b^{nx} - 1}{nx} \\&= m \log a - n \log b \\&= \log \left(\frac{a^m}{b^n} \right)\end{aligned}$$

Q9

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{a^x + b^x + c^x - 3}{x} \\&= \lim_{x \rightarrow 0} \frac{a^x - 1}{x} + \lim_{x \rightarrow 0} \frac{b^x - 1}{x} + \lim_{x \rightarrow 0} \frac{c^x - 1}{x} \\&= \log a + \log b + \log c \\&= \log (abc)\end{aligned}$$

Q10

$$\begin{aligned}\text{Let } x - 2 = h \\ \lim_{h \rightarrow 0} \frac{h}{\log_a(h+1)} \\&= \lim_{h \rightarrow 0} \frac{\log a}{\frac{\log(h+1)}{h}} \\&= \log a\end{aligned}$$

Q11

$$\begin{aligned}
& \lim_{x \rightarrow 0} \frac{5^x + 3^x + 2^x - 3}{x} \\
&= \lim_{x \rightarrow 0} \frac{5^x - 1}{x} + \lim_{x \rightarrow 0} \frac{3^x - 1}{x} + \lim_{x \rightarrow 0} \frac{2^x - 1}{x} \\
&= \log 5 + \log 3 + \log 2 \\
&= \log 30
\end{aligned}$$

Q12

$$\text{Let } \frac{1}{x} = h$$

$$\begin{aligned}
& \lim_{h \rightarrow 0} \frac{(a^h - 1)}{h} \\
&= \log a
\end{aligned}$$

Q13

$$\begin{aligned}
& \lim_{x \rightarrow 0} \frac{a^{mx} - b^{nx}}{\sin kx} \\
&= \lim_{x \rightarrow 0} \frac{a^{mx} - b^{nx}}{kx \frac{\sin kx}{kx}} \\
&= \frac{1}{k} \lim_{x \rightarrow 0} \frac{\frac{(a^{mx} - b^{nx})}{x}}{\frac{\sin kx}{kx}} \\
&= \frac{1}{k} \log \frac{a^m}{b^n}
\end{aligned}$$

Q14

$$\begin{aligned}
& \lim_{x \rightarrow 0} \frac{a^x + b^x - c^x - d^x}{x} \\
&= \lim_{x \rightarrow 0} \frac{a^x - 1}{x} + \lim_{x \rightarrow 0} \frac{b^x - 1}{x} - \lim_{x \rightarrow 0} \frac{c^x - 1}{x} - \lim_{x \rightarrow 0} \frac{d^x - 1}{x} \\
&= \log a + \log b - \log c - \log d \\
&= \log \left(\frac{ab}{cd} \right)
\end{aligned}$$

Q15

$$\begin{aligned}
& \lim_{x \rightarrow 0} \frac{e^x - 1 + \sin x}{x} \\
&= \lim_{x \rightarrow 0} \frac{e^x - 1}{x} + \lim_{x \rightarrow 0} \frac{\sin x}{x} \\
&= \log e + 1 \\
&= 2
\end{aligned}$$

Q16

$$\begin{aligned}
& \lim_{x \rightarrow 0} \frac{\sin 2x}{e^x - 1} \\
&= \lim_{x \rightarrow 0} \frac{\sin 2x}{2x} \times 2 \lim_{x \rightarrow 0} \frac{x}{e^x - 1} \\
&= 1 \times 2 \times \log e \\
&= 2
\end{aligned}$$

Q17

$$\begin{aligned}
& \lim_{x \rightarrow 0} \frac{e^{\sin x} - 1}{x} \\
&= \lim_{x \rightarrow 0} \frac{e^{\sin x} - 1}{\sin x} \times \lim_{x \rightarrow 0} \frac{\sin x}{x} \\
&= \log e \times 1 \\
&= 1
\end{aligned}$$

Q18

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{e^{2x} - e^x}{\sin 2x} \\ &= \lim_{x \rightarrow 0} \frac{e^{2x} - 1}{\sin 2x} - \lim_{x \rightarrow 0} \frac{e^x - 1}{\sin 2x} \\ &= \left(\lim_{x \rightarrow 0} \frac{e^{2x} - 1}{2x} \times \lim_{x \rightarrow 0} \frac{2x}{\sin 2x} \right) - \frac{1}{2} \left(\lim_{x \rightarrow 0} \frac{e^x - 1}{x} \times \lim_{x \rightarrow 0} \frac{2x}{\sin 2x} \right) \\ &= 1 - \frac{1}{2} \\ &= \frac{1}{2} \end{aligned}$$

Q19

$$\begin{aligned} & \lim_{x \rightarrow a} \frac{\log x - \log a}{x - a} \\ &= \lim_{x \rightarrow a} \frac{\log \frac{x}{a}}{a \left(\frac{x}{a} - 1 \right)} \\ & \text{let } h = \frac{x}{a} - 1 \\ &= \frac{1}{a} \lim_{x \rightarrow a} \frac{\log(h+1)}{h} \\ &= \frac{1}{a} \end{aligned}$$

Q20

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{\log(a+x) - \log(a-x)}{x} \\ &= \lim_{x \rightarrow 0} \frac{\log \left(\frac{a+x}{a-x} \right)}{x} \\ &= \lim_{x \rightarrow 0} \frac{\log \left(1 + \frac{2x}{a-x} \right)}{\frac{2x}{a-x}} \times \lim_{x \rightarrow 0} \frac{2}{a-x} \\ &= \frac{2}{a} \end{aligned}$$

Q21

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\log(2+x) + \log(0.5)}{x} \\&= \lim_{x \rightarrow 0} \frac{\log\left(1 + \frac{x}{2}\right)}{2\left(\frac{x}{2}\right)} \\&= \frac{1}{2}\end{aligned}$$

Q22

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\log(a+x) - \log(a)}{x} \\&= \lim_{x \rightarrow 0} \frac{\log\left(1 + \frac{x}{a}\right)}{a\left(\frac{x}{a}\right)} \\&= \frac{1}{a}\end{aligned}$$

Q23

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\log(3+x) - \log(3-x)}{x} \\&= \lim_{x \rightarrow 0} \frac{\log\left(\frac{3+x}{3-x}\right)}{x} \\&= \lim_{x \rightarrow 0} \frac{\log\left(1 + \frac{2x}{3-x}\right)}{x} \\&= \lim_{x \rightarrow 0} \frac{\log\left(1 + \frac{2x}{3-x}\right)}{\frac{2x}{3-x}} \times \lim_{x \rightarrow 0} \frac{2}{3-x} \\&= \frac{2}{3}\end{aligned}$$

Q24

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{8^x - 2^x}{x} \\&= \lim_{x \rightarrow 0} \frac{8^x - 1}{x} - \lim_{x \rightarrow 0} \frac{2^x - 1}{x} \\&= \log 8 - \log 2 \\&= \log 4\end{aligned}$$

Q25

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{x(2^x - 1)}{1 - \cos x} \\&= \lim_{x \rightarrow 0} \frac{x(2^x - 1)}{2\sin^2\left(\frac{x}{2}\right)} \\&= \lim_{x \rightarrow 0} \frac{(2^x - 1)}{x} \times \lim_{x \rightarrow 0} \frac{x^2}{\left(\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}\right)^2 \times \frac{x^2}{2}} \\&= 2\log 2 \\&= \log 4\end{aligned}$$

Q26

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1}{\log(1+x)} \\&= \lim_{x \rightarrow 0} \frac{(\sqrt{1+x} - 1)(\sqrt{1+x} + 1)}{\log(1+x)(\sqrt{1+x} + 1)} \\&= \lim_{x \rightarrow 0} \frac{x}{\log(1+x)(\sqrt{1+x} + 1)} \\&= \lim_{x \rightarrow 0} \frac{1}{\frac{\log(1+x)}{x}} \times \lim_{x \rightarrow 0} \frac{1}{(\sqrt{1+x} + 1)} \\&= 1 \times \frac{1}{2} \\&= \frac{1}{2}\end{aligned}$$

Q27

$$\begin{aligned}
& \lim_{x \rightarrow 0} \frac{\log|1+x^3|}{\sin^3 x} \\
&= \lim_{x \rightarrow 0} \frac{\log|1+x^3|}{\sin^3 x} \times \frac{1}{\lim_{x \rightarrow 0} \left(\frac{\sin x}{x}\right)^3} \\
&= 1 \times 1 \\
&= 1
\end{aligned}$$

Q28

$$\begin{aligned}
& \lim_{x \rightarrow \frac{\pi}{2}} \frac{a^{\cot x} - a^{\cos x}}{\cot x - \cos x} \\
&= \lim_{x \rightarrow \frac{\pi}{2}} a^{\cos x} \left[\frac{a^{\cot x - \cos x} - 1}{\cot x - \cos x} \right] \\
&= 1 \times \log a \\
&= \log a
\end{aligned}$$

Q29

$$\begin{aligned}
& \lim_{x \rightarrow 0} \frac{e^x - 1}{\sqrt{1 - \cos x}} \\
&= \lim_{x \rightarrow 0} \frac{(e^x - 1)(\sqrt{1 + \cos x})}{(\sqrt{1 - \cos x})(\sqrt{1 + \cos x})} \\
&= \lim_{x \rightarrow 0} \frac{(e^x - 1)(\sqrt{1 + \cos x})}{\sin x}
\end{aligned}$$

Both numerator and denominator are both zeros for $x = 0$
hence limit can not exist

Q30

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{e^{5+h} - e^5}{h} \\&= e^5 \lim_{x \rightarrow 0} \frac{e^h - 1}{h} \\&= e^5 \times 1 \\&= e^5\end{aligned}$$

Q31

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{e^{x+2} - e^2}{x} \\&= e^2 \lim_{x \rightarrow 0} \frac{e^x - 1}{x} \\&= e^2\end{aligned}$$

Q32

$$\begin{aligned}\text{Let } x - \frac{\pi}{2} = h \\ \lim_{h \rightarrow 0} \frac{e^{-\sin h} - 1}{-\sin h} \\&= \lim_{\sin h \rightarrow 0} \frac{e^{-\sin h} - 1}{-\sin h} \\&= 1\end{aligned}$$

Q33

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{e^{3+x} - \sin x - e^3}{x} \\&= e^3 \lim_{x \rightarrow 0} \frac{e^x - 1}{x} - \lim_{x \rightarrow 0} \frac{\sin x}{x} \\&= e^3 \log e - 1 \\&= e^3 - 1\end{aligned}$$

Q34

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{e^x - x - 1}{x} \\&= \lim_{x \rightarrow 0} \frac{e^x - 1}{x} - 1 \\&= 1 - 1 \\&= 0\end{aligned}$$

Q35

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{e^{3x} - e^{2x}}{x} \\&= 3 \lim_{x \rightarrow 0} \frac{e^{3x} - 1}{3x} - \lim_{x \rightarrow 0} \frac{e^{2x} - 1}{2x} \\&= 3 - 2 \\&= 1\end{aligned}$$

Q36

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{e^{\tan x} - 1}{\tan x} \\&= \lim_{\tan x \rightarrow 0} \frac{e^{\tan x} - 1}{\tan x} \\&= 1\end{aligned}$$

Q37

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{e^{bx} - e^{ax}}{x} \\&= b \lim_{x \rightarrow 0} \frac{e^{bx} - 1}{bx} - a \lim_{x \rightarrow 0} \frac{e^{ax} - 1}{ax} \\&= b - a\end{aligned}$$

Q38

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{e^{\tan x} - 1}{x} \\&= \lim_{x \rightarrow 0} \frac{e^{\tan x} - 1}{\tan x} \times \lim_{x \rightarrow 0} \frac{\tan x}{x} \\&= \log e \times 1 \\&= 1\end{aligned}$$

Q39

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{e^x - e^{\sin x}}{x - \sin x} \\&= \lim_{x \rightarrow 0} e^{\sin x} \left[\frac{e^{x - \sin x} - 1}{x - \sin x} \right] \\&= 1 \times \log e \\&= 1\end{aligned}$$

Q40

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{3^{2+x} - 9}{x} &= \lim_{x \rightarrow 0} \frac{3^2 \cdot 3^x - 9}{x} \\&= 9 \lim_{x \rightarrow 0} \frac{3^x - 1}{x} \\&= 9 \log_e 3\end{aligned}$$

Q41

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{a^x - a^{-x}}{x} &= \lim_{x \rightarrow 0} \frac{a^{2x} - 1}{x a^x} \\&= \lim_{x \rightarrow 0} \frac{2(a^{2x} - 1)}{2x} \lim_{x \rightarrow 0} \frac{1}{a^x} \\&= 2 \log_e 2\end{aligned}$$

Q42

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{x(e^x - 1)}{1 - \cos x} \\&= \lim_{x \rightarrow 0} \frac{x(e^x - 1)}{2\sin^2\left(\frac{x}{2}\right)} \\&= \lim_{x \rightarrow 0} \frac{(e^x - 1)}{2x} \times \lim_{x \rightarrow 0} \frac{4}{\left(\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}\right)^2} \\&= \frac{1}{2} \times 4 \\&= 2\end{aligned}$$

Q43

$$\begin{aligned}\lim_{x \rightarrow \frac{\pi}{2}} \frac{2^{-\cos x} - 1}{x\left(x - \frac{\pi}{2}\right)} &= \lim_{x \rightarrow \frac{\pi}{2}} \frac{2^{\sin\left(x - \frac{\pi}{2}\right)} - 1}{\left(x - \frac{\pi}{2}\right)} \times \frac{1}{x} \\&= \frac{2}{\pi} \log_e 2\end{aligned}$$

Ex 29.11

Q1

$$\begin{aligned}\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n &= e^{\lim_{n \rightarrow \infty} \left(\frac{x}{n}\right)n} \\ &= e^x\end{aligned}$$

Q2

$$\begin{aligned}&\lim_{x \rightarrow 0^+} \left\{1 + \tan^2 \sqrt{x}\right\}^{1/2x} \\ &= e^{\lim_{x \rightarrow 0^+} \left\{\frac{\tan^2 \sqrt{x}}{2x}\right\}} \\ &= e^{\lim_{x \rightarrow 0^+} \left\{\frac{\tan^2 \sqrt{x}}{2x}\right\}} \\ &= e^{\lim_{x \rightarrow 0^+} \left\{\frac{\sin^2 \sqrt{x}}{2x \cos^2 \sqrt{x}}\right\}} \\ &= e^{\frac{1}{2} \lim_{x \rightarrow 0^+} \left\{\left(\frac{\sin \sqrt{x}}{\sqrt{x}}\right)^2\right\} \lim_{x \rightarrow 0^+} \left\{\frac{1}{\cos^2 \sqrt{x}}\right\}} \\ &= e^{\frac{1}{2}} \\ &= \sqrt{e}\end{aligned}$$

Q3

$$\begin{aligned}
 \lim_{x \rightarrow 0} (\cos x)^{1/\sin x} &= \lim_{x \rightarrow 0} (1 + \cos x - 1)^{1/\sin x} \\
 &= \lim_{x \rightarrow 0} (1 - (1 - \cos x))^{1/\sin x} \\
 &= \lim_{x \rightarrow 0} \left(1 - 2 \sin^2 \left(\frac{x}{2} \right) \right)^{1/\sin x} \\
 &= e^{\lim_{x \rightarrow 0} \left(-2 \sin^2 \left(\frac{x}{2} \right) \right) \times (1/\sin x)} \\
 &= e^{\lim_{x \rightarrow 0} \left(\frac{-2 \sin^2 \left(\frac{x}{2} \right)}{\sin x} \right)} \\
 &= e^{\lim_{x \rightarrow 0} \left(\frac{-2 \sin^2 \left(\frac{x}{2} \right)}{2 \sin \left(\frac{x}{2} \right) \cos \left(\frac{x}{2} \right)} \right)} \\
 &= e^{\lim_{x \rightarrow 0} -\tan x} \\
 &= e^0 \\
 &= 1
 \end{aligned}$$

Q4

$$\begin{aligned}
 \lim_{x \rightarrow 0} (\cos x + \sin x)^{1/x} &= \lim_{x \rightarrow 0} (1 + (\cos x + \sin x - 1))^{1/x} \\
 &= e^{\lim_{x \rightarrow 0} \frac{(\cos x + \sin x - 1)}{x}} \\
 &= e^{\lim_{x \rightarrow 0} \frac{(\sin x - (1 - \cos x))}{x}} \\
 &= e^{\lim_{x \rightarrow 0} \frac{(\sin x - 2 \sin^2(x/2))}{x}} \\
 &= e^{\lim_{x \rightarrow 0} \frac{\sin x}{x} - \lim_{x \rightarrow 0} \frac{2 \sin(x/2) \sin(x/2)}{2 \left(\frac{x}{2} \right)}} \\
 &= e^{\lim_{x \rightarrow 0} \frac{\sin x}{x} - \lim_{x \rightarrow 0} \frac{\sin(x/2) \sin(x/2)}{\left(\frac{x}{2} \right)}} \\
 &= e^{1-0} \\
 &= e
 \end{aligned}$$

Q5

$$\begin{aligned}
 & \lim_{x \rightarrow 0} (\cos x + a \sin bx)^{1/x} \\
 &= \lim_{x \rightarrow 0} (1 + (\cos x + a \sin bx - 1))^{1/x} \\
 &= e^{\lim_{x \rightarrow 0} \frac{(\cos x + a \sin bx - 1)}{x}} \\
 &= e^{\lim_{x \rightarrow 0} \frac{(a \sin bx - (1 - \cos x))}{x}} \\
 &= e^{\lim_{x \rightarrow 0} \frac{(a \sin bx - 2 \sin^2(x/2))}{x}} \\
 &= e^{\lim_{x \rightarrow 0} \frac{ab \sin bx}{bx} - \lim_{x \rightarrow 0} \frac{2 \sin(x/2) \sin(x/2)}{2 \left(\frac{x}{2}\right)}} \\
 &= e^{\lim_{x \rightarrow 0} \frac{ab \sin bx}{bx} - \lim_{x \rightarrow 0} \frac{\sin(x/2) \sin(x/2)}{\left(\frac{x}{2}\right)}} \\
 &= e^{ab - 0} \\
 &= e^{ab}
 \end{aligned}$$

Q6

$$\begin{aligned}
 & \lim_{x \rightarrow \infty} \left\{ \frac{x^2 + 2x + 3}{2x^2 + x + 5} \right\}^{\frac{3x-2}{3x+2}} \\
 &= e^{\lim_{x \rightarrow \infty} \left\{ \left(\frac{3x-2}{3x+2} \right) \ln \left(\frac{x^2 + 2x + 3}{2x^2 + x + 5} \right) \right\}} \\
 &= e^{\lim_{x \rightarrow \infty} \left\{ \left(\frac{3 - \frac{2}{x}}{3 + \frac{2}{x}} \right) \left(\ln \left(\frac{x^2 + 2x + 3}{2x^2 + x + 5} \right) \right) \right\}} \\
 &= e^{\lim_{x \rightarrow \infty} \left\{ \left(\frac{3 - \frac{2}{x}}{3 + \frac{2}{x}} \right) \left(\ln \left(\frac{1 + \frac{2}{x} + \frac{3}{x^2}}{2 + \frac{1}{x} + \frac{5}{x^2}} \right) \right) \right\}} \\
 &= e^{1 \ln \left(\frac{1}{2} \right)} \\
 &= \frac{1}{2}
 \end{aligned}$$

Q7

$$\begin{aligned}& \lim_{x \rightarrow 1} \left\{ \frac{x^3 + 2x^2 + x + 1}{x^2 + 2x + 3} \right\}^{\frac{1 - \cos(x-1)}{(x-1)^2}} \\&= e^{\lim_{x \rightarrow 1} \left\{ \frac{1 - \cos(x-1)}{(x-1)^2} \ln \left(\frac{x^3 + 2x^2 + x + 1}{x^2 + 2x + 3} \right) \right\}} \\&= e^{\lim_{x \rightarrow 1} \left\{ \frac{2 \sin^2 \left(\frac{x-1}{2} \right)}{4 \left(\frac{x-1}{2} \right)^2} \ln \left(\frac{x^3 + 2x^2 + x + 1}{x^2 + 2x + 3} \right) \right\}} \\&= e^{\frac{2}{4} \ln \left(\frac{5}{6} \right)} \\&= e^{\ln \left(\frac{5}{6} \right)^{\frac{1}{2}}} \\&= \left(\frac{5}{6} \right)^{\frac{1}{2}} = \sqrt{\frac{5}{6}}\end{aligned}$$

Q8

$$\lim_{x \rightarrow 0} \left\{ \frac{e^x + e^{-x} - 2}{x^2} \right\}^{1/x^2}$$

$$= e^{\lim_{x \rightarrow 0} \frac{\left\{ \frac{e^x + e^{-x} - 2}{x^2} \right\}}{x^2}}$$

Applying L'Hospital's Rule

$$= e^{\lim_{x \rightarrow 0} \left\{ \frac{2 + e^x(-2+x) + x}{2(-1+e^x)x^2} \right\}}$$

Applying L'Hospital's Rule again

$$= e^{\frac{1}{2} \lim_{x \rightarrow 0} \left\{ \frac{1 + e^x(-1+x)}{x(-2 + e^x(2+x))} \right\}}$$

Applying L'Hospital's Rule again

$$= e^{\frac{1}{2} \lim_{x \rightarrow 0} \left\{ \frac{e^x x}{-2 + e^x(2+4x+x^2)} \right\}}$$

Applying L'Hospital's Rule again

$$= e^{\frac{1}{2} \lim_{x \rightarrow 0} \left\{ \frac{1+x}{6+6x+x^2} \right\}}$$

$$= e^{\frac{1}{2} \left\{ \frac{\lim_{x \rightarrow 0} (1+x)}{\lim_{x \rightarrow 0} (6+6x+x^2)} \right\}}$$

$$= e^{\frac{1}{12}}$$

$$= \sqrt[12]{e}$$

Q9

$$\begin{aligned}& \lim_{x \rightarrow a} \left\{ \frac{\sin x}{\sin a} \right\}^{\frac{1}{x-a}} \\&= \lim_{x \rightarrow a} \left\{ 1 + \left(\frac{\sin x}{\sin a} - 1 \right) \right\}^{\frac{1}{x-a}} \\&= e^{\lim_{x \rightarrow a} \left\{ \frac{\left(\frac{\sin x}{\sin a} - 1 \right)}{x-a} \right\}} \\&= e^{\lim_{x \rightarrow a} \left\{ \frac{\left(\frac{\sin x - \sin a}{\sin a} \right)}{x-a} \right\}} \\&= e^{\lim_{x \rightarrow a} \left\{ \frac{\sin x - \sin a}{\sin a (x-a)} \right\}} \\&= e^{\lim_{x \rightarrow a} \left\{ \frac{2 \cos \left(\frac{x+a}{2} \right) \sin \left(\frac{x-a}{2} \right)}{\sin a (x-a)} \right\}} \\&= e^{\lim_{x \rightarrow a} \left\{ \frac{2 \cos \left(\frac{x+a}{2} \right)}{\sin a} \right\} \lim_{x \rightarrow a} \left\{ \frac{\sin \left(\frac{x-a}{2} \right)}{2 \left(\frac{x-a}{2} \right)} \right\}} \\&= e^{\frac{2 \cos a}{2 \sin a}} \\&= e^{\cot a}\end{aligned}$$

Q10

$$\begin{aligned}& \lim_{x \rightarrow \infty} \left\{ \frac{3x^2 + 1}{4x^2 - 1} \right\}^{\frac{x^3}{1+x}} \\&= \lim_{x \rightarrow \infty} \left\{ 1 + \frac{-x^2 + 2}{4x^2 - 1} \right\}^{\frac{x^3}{1+x}} \\&= e^{\lim_{x \rightarrow \infty} \left\{ \left(\frac{-x^2 + 2}{4x^2 - 1} \right) \left(\frac{x^3}{1+x} \right) \right\}} \\&= e^{\lim_{x \rightarrow \infty} \left\{ \left(\frac{-x^5 + 2x^3}{4x^2 - 1 + 4x^3 - x} \right) \right\}} \\&= e^{-\infty} \\&= 0\end{aligned}$$